

Expansion Method for Solving Fredholm Integral Equations in Space-time

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Abstract

In this paper a new method for solving the integral equation in space-time which arises in mathematical physics, mechanics and the heat conduction theory is presented. The central idea is: First, differentiating both sides of two-dimensions Fredholm integral Equations n-times with respect to both variables x and t, second, substituting the Taylor series of two variables for the unknown function, and third, obtain a system of linear equations which can be solved by a suitable truncation scheme .Lastly, the double-quadrature rules are used to calculate the required integrals in this procedure. The presented algorithm is illustrated by some numerical examples with comparison tables.

Introduction

Consider the following linear two-dimensional Fredholm integral equations of second kind:

$$f(x,t) = g(x,t) + \int\int_{c \ a}^{d \ b} k(x,t,y,s) f(y,s) dy ds \quad \dots(1)$$

For $(x,t) \in D := [a,b] \times [c,d]$, where $k : D \times D \rightarrow R$ and $g : D \rightarrow R$ are assumed to be bounded and usually to be continuous, the two-variable function f is the unknown function, while a,b,c and d are real constants(Vasile,2001). This equation has a unique solution in the space $L^2(D)$, (Lechoslow,2002).The considered integral equations in space-time play a very important role in mechanics and technology, with special attention paid to large sense of power engineering. Some initial-boundary problems for a number of differential partial equations in physics can be reduced to consider integral equation, (Lechoslow,2002).Numerical results for equation (1) have been treated by many authors and different methods are used, by:Valise Carutasu (spline functions), Lechoslaw Hacia (Galerkin method) and Brunner (collocation method), (Vasile,2001,Lechoslow,2002,Brunner&Kurtis,1989).Fewer numerical and expansion methods are known for the several-dimensional Fredholm integral equations.In this paper we will deal with the Taylor expansion method for the two-dimensional linear Fredholm integral equations of second kind.

Taylor series (John&Kurtis,2004,Richard&Douglas,1997)

Suppose $f(x, y)$ and its partial derivation through order $(m+1)$ are continuous throughout a rectangular region centered at a point (x_0, y_0) . Then throughout the region,

$$f(x, y) = f(x_0, y_0) + (h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y})f(x_0, y_0) + \frac{1}{2!}(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y})^2 f(x_0, y_0) + \dots \dots (2)$$

$$+ \frac{1}{m!}(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y})^m f(x_0, y_0) + R_m$$

Where R_m , the remainder after m-terms, is given by:

$$R_m = \frac{1}{(m+1)!} (h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y})^{m+1} f(x_0 + \theta h, y_0 + \theta k) , 0 < \theta < 1$$

where $h = \Delta x = x - x_0$, $k = \Delta y = y - y_0$ and the operator notation is,

$$(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y})^2 f(x_0, y_0) = h^2 f_{xx}(x_0, y_0) + 2hk f_{xy}(x_0, y_0) + k^2 f_{yy}(x_0, y_0)$$

And if we expanded m-th term formally by the binomial theorem, we get

$$(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y})^m f(x_0, y_0) = \sum_{i=0}^m \binom{m}{i} h^{m-i} k^i \frac{\partial^i}{\partial y^i} \left(\frac{\partial^{m-i} f(x_0, y_0)}{\partial x^{m-i}} \right)$$

Finally we substitute this result in the equation (2) and we obtain the following equation

$$f(x, y) = \sum_{m=0}^{\infty} \sum_{i=0}^m \frac{1}{m!} \binom{m}{i} (x - x_0)^{m-i} (y - y_0)^i \frac{\partial^i}{\partial y^i} \left(\frac{\partial^{m-i} f(x_0, y_0)}{\partial x^{m-i}} \right) \dots (3)$$

The method

In order to solve equ.(1) we take the partial derivatives n-times with respective to x and t respectively, and get :

$$\frac{\partial^n f(x, t)}{\partial t^r \partial x^{n-r}} = \frac{\partial^n g(x, t)}{\partial t^r \partial x^{n-r}} + \iint_{c a}^{d b} \frac{\partial^n k(x, t, y, s)}{\partial t^r \partial x^{n-r}} f(y, s) dy ds , \text{ for each } r = 0, 1, \dots, n$$

Replace (x, t) by $(x_0, t_0) \in D$ in the last equation, we have

$$\frac{\partial^n f(x_0, t_0)}{\partial t^r \partial x^{n-r}} = \frac{\partial^n g(x_0, t_0)}{\partial t^r \partial x^{n-r}} + \iint_{c a}^{d b} \frac{\partial^n k(x_0, t_0, y, s)}{\partial t^r \partial x^{n-r}} f(y, s) dy ds \dots (4)$$

Then expand $f(y, s)$ in Taylor series of two-variables at (x_0, t_0) as in equ.(3) and substituting it in equ.(4) to obtain :

$$\frac{\partial^n f(x_0, t_0)}{\partial t^r \partial x^{n-r}} = \frac{\partial^n g(x_0, t_0)}{\partial t^r \partial x^{n-r}} +$$

$$\iint_{c a}^{d b} \frac{\partial^n k(x_0, t_0, y, s)}{\partial t^r \partial x^{n-r}} \left[\sum_{m=0}^{\infty} \sum_{i=0}^m \frac{1}{m!} \binom{m}{i} (y - x_0)^{m-i} (s - t_0)^i \frac{\partial^i}{\partial s^i} \left(\frac{\partial^{m-i} f(x_0, t_0)}{\partial y^{m-i}} \right) \right] dy ds$$

And we rewrite it, since by substituting the derivative part of f and g at the point (x_0, t_0) so we replace x and t in the derivative function f and g by y and s respectively which are the same and we get:

$$\frac{\partial^n f(x_0, t_0)}{\partial s^r \partial y^{n-r}} = \frac{\partial^n g(x_0, t_0)}{\partial s^r \partial y^{n-r}} + \sum_{m=0}^{\infty} \sum_{i=0}^m \frac{\partial^i}{\partial s^i} \left(\frac{\partial^{m-i} f(x_0, t_0)}{\partial y^{m-i}} \right) T_{nrm} \quad \dots(5)$$

Where

$$T_{nrm} = \frac{1}{i!(m-i)!} \int_c^d \int_a^b \frac{\partial^n k(x_0, t_0, y, s)}{\partial t^r \partial x^{n-r}} (y - x_0)^{m-i} (s - t_0)^i dy ds \quad , \text{for each } r = 0, 1, \dots, n$$

Finally, we will attempt to solve the infinite relations from equ.(5) to determining the Taylor coefficients by a suitable technique which depends on truncating the series in such way that we get a determinate system at every step. Equation (5) can be expressed in matrix form as:

$$G = HF \quad \dots(6)$$

Where H and G are constant matrices with dimensions $p \times p$ and $p \times 1$ respectively, and F is a matrix of Taylor's coefficients with dimension $p \times 1$, such that: $p = \frac{(m+1)(m+2)}{2}, m = 0, 1, 2, \dots$; Where

$$F = [f_q(x_0, t_0)]^T \quad \text{and} \quad G = [g_q(x_0, t_0)]^T \quad \text{for each } q = 1, 2, \dots, p$$

On the other hand, $q = ij$ (as a component) where

$j = 0, 1, \dots, m$ and $i = 0, 1, \dots, j$ with

$$f_q = f_{ij} = \frac{\partial^i}{\partial s^i} \left(\frac{\partial^{j-i} f}{\partial y^{j-i}} \right) \quad \text{and} \quad g_q = g_{ij} = \frac{\partial^i}{\partial s^i} \left(\frac{\partial^{j-i} g}{\partial y^{j-i}} \right)$$

Furthermore:

$$H = [A_{\ell i}] \quad \text{and} \quad A_{\ell i} = [T_{\ell r i j}]$$

where $\ell = 0, 1, \dots, n$; $r = 0, 1, \dots, \ell$

and $i = 0, 1, \dots, m$; $j = 0, 1, \dots, i$

At last, subtract one from each element of the diagonal in matrix H ($\ell = i$ & $r = j$) so becomes $(T_{\ell r i j} - 1)$, for more detail see Appendix A.

A computation efficient way to calculate the linear algebraic systems of coefficients for equ.(5) is to store the matrix H and compute $H^T H$ and $H^T G$ in equ.(6), if $n > m$, then we use any procedure to solve the linear system :

$$H^T G = H^T H F$$

While if $n = m$ we use directly the methods for solving equ.(6) . We discuss the case where $n = m$, substitute the values f_q ’s in equ.(3),we get the approximation solution $\hat{f}(x, y)$ of equ.(1).Then the resulting method error $e(x, y) = f(x, y) - \hat{f}(x, y)$ satisfies:

$$\|e\|_{\infty} = \sup\{|e(x, y)| : (x, y \in D)\} \text{ (as } m, n \rightarrow \infty\text{), (Vasile,2001)}$$

Because the calculation of error, given in above, is not easy so we use the norm of matrix to study the quantitatively of the error in this approximation method, and we use the relative error which is $\|\hat{f} - f\|/\|f\|$,(John & Kurtis,2004).

The method presented above was implemented in the MATLAB6.5, (see Appendix B).

Numerical tests

Now we illustrate our method by given three numerical examples the exact solutions of which are known:

Example(1):

Consider the following integral equation

$$f(x, t) = \frac{1}{3}(1 - 2x \cos(t)) - x^2 + t + \int_0^1 \int_0^1 (x \cos(t) - sy) f(y, s) dy ds, \quad (x, t) \in [0,1] \times [0,1]$$

$$\text{So that: } k(x, t, y, s) = x \cos(t) - sy \quad \text{and} \quad g(x, t) = \frac{1}{3}(1 - 2x \cos(t)) - x^2 + t$$

To process equation above we evaluate the quantities T_{nrm} and $\frac{\partial^n g}{\partial s^r \partial y^{n-r}}$

at $(x_0, t_0) = (0,0)$ and take $n = m = 2$, so we obtain

$$H = [T_{nrm}] = \begin{bmatrix} -3/4 & 1/6 & 1/6 & 1/16 & 1/9 & 1/16 \\ 1 & -1/2 & 1/2 & 1/6 & 1/4 & 1/6 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

And

$$G = -\left[\frac{\partial^n g}{\partial s^r \partial y^{n-r}} \right]^T = -[1/3 \quad -2/3 \quad 1 \quad -2 \quad 0 \quad 0]^T$$

Next, we substitute these values in equ.(6) then getting the values of F as:

$$F = [1/2 \quad 0 \quad 1 \quad -2 \quad 0 \quad 0]^T$$

Which are the coefficients of Taylor's series. Putting these values in Taylor expansion (3) we obtain $\hat{f}(x,t) = 0.5 + t - x^2$ which is the exact solution.

We could do all these manipulations in program, by writing it for this propose, which can find the relative error for it with Running Time (R.T.). In this example Error=0.0 and R.T=0.844.

Example (2):

Consider the following double-Fredholm integral equation

$$f(x,y) = x + \sin(t) - \frac{2}{3}e^x + \int_0^1 \int_0^1 ye^x f(y,s) dy ds \quad (x,t) \in [-1,1] \times [0,1]$$

whose exact solution is:

$$f(x,t) = x + \sin(t)$$

The exact solution will be approximated by $\hat{f}(x,t)$, by Taylor's method in section-3. Take

$m = n = 3$ with $(x_0, t_0) = (0,0)$, after obtaining H and G substituting them in equ.(6) , then

we get the value of F as : $F = [0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ -1]^T$

Which are the coefficients of Taylor's series. Putting these values in equ.(3) to get:

$$\hat{f}(x,t) = x + t - \frac{1}{3!}t^5$$

Table (1) presents a comparison between the exact and approximate solution which depends on relative error and running time with different value of m and n .

$n=m$	Approximation solution	Error	R.T
3	$\hat{f}(x,t) = x + t - 1.666667t^3$	0.003315	1.844
5	$\hat{f}(x,t) = x + t - 1.666667t^3 + 0.008333t^5$	6.86637×10^{-5}	6.906
7	$\hat{f}(x,t) = x + t - 1.666667t^3 + 0.008333t^5 - 0.000198t^7$	1.01465×10^{-6}	19.328
9	$\hat{f}(x,t) = x + t - 1.666667t^3 + 0.008333t^5 - 0.000198t^7 + 0.0000027t^9$	1.63335×10^{-7}	44.781

Example (3):

Consider the problem

$$f(x, y) = t \cos(x) - \frac{\cos(1) + \sin(1) - 1}{4} x^2 e^t + \int_0^1 \int_0^1 s^2 x^2 y e^t f(y, s) dy ds \quad (x, t) \in [0,1] \times [0,1]$$

whose exact solution is:

$$f(x, t) = t \cos(x)$$

The exact solution will be approximated by $\hat{f}(x, t)$, by Taylor's method in section (3).Take

$m = n = 2$ with $(x_0, t_0) = (0.5, 0.5)$, after obtaining H and G substituting them in equ.(6) , then we get the value of F as :

$$F = [0.439396 \quad -0.237292 \quad 0.878788 \quad -0.433950 \quad -0.477005 \quad 0.60511 \times 10^{-3}]^T$$

Which are the coefficients of Taylor's series. Putting these values in equ.(3) to get:

$$\begin{aligned} \hat{f}(x, t) = & 0.118948 - 0.237292x + 0.878188t - 0.216975(x - 0.5)^2 \\ & - 0.477005(x - 0.5)(t - 0.5) + 0.302555 \times 10^{-3}(t - 0.5)^2 \end{aligned}$$

Table (2) presents a comparison between the exact and approximate solution which depends on relative error and running time with different value of m and n .

n=m	Error	R.T
2	0.029477	1.438
3	0.002400	3.218
4	4.70887×10^{-4}	6.547
5	2.39951×10^{-5}	10.750
6	3.66377×10^{-6}	18.078

Conclusions

In this paper, we have introduced a new numerical method for approximating the solution of two-dimensional Fredholm integral equations, in which a Taylor expansion is used. Several examples were applied for illustration and good results achieved. We conclude the following points:

1.The good approximation depends on:

- i. Increasing the number of terms taken in Taylor method (see table (1&2)).
- ii.The good estimate of T_{nrm_i} in equ.(5) which is made by using suitable quadrature method for double integration.

- 2.The method described here should work reasonably well on problems where the unknown function f is assumed to be a polynomial and get the exact solution in many cases.
- 3.The disadvantage here, when we take n -large number we get very large square matrix H in equ. (6) with dimension $\frac{(n+1)(n+2)}{2} \times \frac{(n+1)(n+2)}{2}$, which need large memory of computer with too much time to compute it.

References

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طريقة التوسيع لحل معادلات فريدهولم التكاملية- المضاعفة

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الخلاصة

أعطينا في هذا البحث طريقة جديدة لحل المعادلات التكاملية المضاعفة التي تظهر في الفيزياء الرياضية، كالميكانيكا ونظرية التوصيل الحراري. أن الفكرة الأساسية في هذا البحث هي:
أولاً: أجري عملية التفاضل لطريقي معادلة فريدهولم التكاملية ذات البعدين $L - (n)$ من المرات بالنسبة للمتغيرين x و t .
ثانياً: تعويض متسلسلة تيلور لمتغيرين مكان الدالة المجهولة $(f(x,t))$.
ثالثاً: نحصل على نظام من المعادلات الخطية والتي يمكن حلها بالطرق المختلفة (ال المناسبة).
وأخيراً تم استخدام الطريقة التربيعية لحساب التكاملات المضاعفة المطلوبة في هذا البحث وكذلك تم إيضاح الخوارزمية المذكورة أعلاه ببعض الأمثلة العددية مع أعطاء جداول مقارنة.

Appendix (A)

For instance, in equation (5) let us take:

(i) $n = 0,1$ & $m = 1$ and obtain

$$\begin{aligned}
 n = 0 ; m = 1 \text{ so } r = 0 \quad f(x_0, t_0) &= g(x_0, t_0) + f(x_0, t_0)T_{0000} + \frac{\partial f(x_0, t_0)}{\partial y}T_{0010} + \frac{\partial f(x_0, t_0)}{\partial s}T_{0011} \\
 n = 1 ; m = 1 \text{ so } r = 0 \quad \frac{\partial f(x_0, t_0)}{\partial y} &= \frac{\partial g(x_0, t_0)}{\partial y} + f(x_0, t_0)T_{1000} + \frac{\partial f(x_0, t_0)}{\partial y}T_{1010} + \frac{\partial f(x_0, t_0)}{\partial s}T_{1011} \\
 r = 1 \quad \frac{\partial f(x_0, t_0)}{\partial s} &= \frac{\partial g(x_0, t_0)}{\partial s} + f(x_0, t_0)T_{1100} + \frac{\partial f(x_0, t_0)}{\partial y}T_{1110} + \frac{\partial f(x_0, t_0)}{\partial s}T_{1111}
 \end{aligned}$$

Rewrite equation above in matrix form as: $G = HF$

$$- \begin{bmatrix} g(x_0, t_0) \\ g_y(x_0, t_0) \\ g_s(x_0, t_0) \end{bmatrix} = \begin{bmatrix} (T_{0000} - 1) & T_{0010} & T_{0011} \\ T_{1000} & (T_{1010} - 1) & T_{1011} \\ T_{1100} & T_{1110} & (T_{1111} - 1) \end{bmatrix} \begin{bmatrix} f(x_0, t_0) \\ f_y(x_0, t_0) \\ f_s(x_0, t_0) \end{bmatrix}$$

(ii) $n = 0,1,2$ & $m = 2$ we obtain:

$$- \begin{bmatrix} g(x_0, t_0) \\ g_y(x_0, t_0) \\ g_s(x_0, t_0) \\ g_{yy}(x_0, t_0) \\ g_{ys}(x_0, t_0) \\ g_{ss}(x_0, t_0) \end{bmatrix} = \begin{bmatrix} (T_{0000} - 1) & T_{0010} & T_{0011} & T_{0020} & T_{0021} & T_{0022} \\ T_{1000} & (T_{1010} - 1) & T_{1011} & T_{1020} & T_{1021} & T_{1022} \\ T_{1100} & T_{1110} & (T_{1111} - 1) & T_{1120} & T_{1121} & T_{1122} \\ T_{2000} & T_{2010} & T_{2011} & (T_{2020} - 1) & T_{2021} & T_{2022} \\ T_{2100} & T_{2110} & T_{2111} & T_{2120} & (T_{2121} - 1) & T_{2122} \\ T_{2200} & T_{2210} & T_{2211} & T_{2220} & T_{2221} & (T_{2222} - 1) \end{bmatrix} \begin{bmatrix} f(x_0, t_0) \\ f_y(x_0, t_0) \\ f_s(x_0, t_0) \\ f_{yy}(x_0, t_0) \\ f_{ys}(x_0, t_0) \\ f_{ss}(x_0, t_0) \end{bmatrix}$$

Appendix (B)

In this appendix we introduce a program in MATLAB6.5 for the method which is given in section-3:

```
clc
format long
```

```
syms x t y s
```

```
% [a,b];[c,d] is the boundary points of integrals
```

```
a=0 ; b=1 ; c=0 ; d=1 ;
```

```
% [x0,t0] is the around points of Taylor's method
```

```
x0=0.5 ; t0=0.5;
```

```
% [n,m] is the number of terms in Taylor expansion
```

```
n=3; m=3 ;
```

```
% tic & toc is using for determine the time in program
tic
q=(n+1)*(n+2)/2;
tt=sym(zeros(q)); g1=sym(zeros(q,1));
L=1;
for i=0:n
    for j=0:i
        w=i-j;
        [z,gg]=pro1(x0,t0,w,j);
        LL=1;
        for i1=0:m
            for j1=0:i1
                w1=i1-j1;
                z11=((y-x0)^w1)*((s-t0)^j1);
                w11=factorial(w1)*factorial(j1);
                zz=z11/w11;
                ss=z*zz;
                t11=int(int(ss,y,a,b),s,c,d);
                if (L==LL)
                    tt(L,LL)=t11-1;
                else
                    tt(L,LL)=t11;
                end
                LL=LL+1;
            end
        end
        g1(L,1)=gg;
        L=L+1;
    end
end
c1=tt\g1;
cq=vpa(c1,6);
Ls=1;f1=0;
for i=0:n
    for j=0:i
        w=i-j;
        w4=factorial(w)*factorial(j);
        f1=f1+(cq(Ls)*(x-x0)^w*(t-t0)^j)/w4;
        Ls=Ls+1; end
end
```

toc

```
% The following steps using to find relative error's by matrix-norm(2)
f=vpa(f1,6),p1q=char(f);apq=inline(p1q,'x','t');
exact=t*cos(x) ,ex=char(exact);exa=inline(ex,'x','t');
n0=100;m0=100;
DF=zeros(n0,m0);DD=zeros(n0,m0);
h0=(b-a)/(n0-1);k10=(d-c)/(m0-1);
for i0=1:n0
    xo=a+(i0-1)*h0;
    for j0=1:m0
        t10=c+(j0-1)*k10;
        DD(i0,j0)=exa(xo,t10);
        DF(i0,j0)=apq(xo,t10);
    end
end
Error=norm(DD-DF)/norm(DD),
function [kg,gk]=pro1(x0,t0,w,j)
format long
syms x t y s
k=s^2*x^2*y*exp(t) ;
k1=diff(k,'x',w);k2=diff(k1,'t',j);
k3=char(k2);
z1=inline(k3,'x','t','y','s');
kg=z1(x0,t0,y,s);
g=t*cos(x)-1/4*x^2*exp(t)*cos(1)-1/4*x^2*exp(t)*sin(1)+1/4*x^2*exp(t) ;
g2=diff(g,'x',w);
g3=diff(g2,'t',j);
g4=char(g3);
g5=inline(g4,'x','t');
gk=-g5(x0,t0);
```