

On The Solution of Certain Fractional Integral Equations

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Abstract

In this paper we introduce the linear operator of fractional integral equation of the second kind (FIESK) in the framework of the Riemann-Liouville fractional calculus. Some results concerning the existence and uniqueness have been also obtained. Particular attention is devoted to the technique of Laplace transform for treating FIESK. By applying this technique we shall derive the analytical solutions of the most linear FIESK. Other main objective concern here is to give an approximate scheme using collocation method to solve FIESK. Two fundamental questions concerning this method: its stability and convergence are discussed. We show that the analytical stability bounds are in excellent agreement with numerical tests. Comparison between exact solutions and approximate predictions is made.

Introduction

Fractional calculus have been a highly specialized and isolated field of mathematics for many years. However, in the last decade there have been increasing interest in the description of physical and chemical processes by means of equations involving fractional derivatives and integrals. This mathematical technique has a board potential range of applications (Delves & Walsh, 1997, Faycal & Jacky 2005, Irmak & Raina, 2004, Ortigueira., 2000, Wheeler, 1997). In recent years considerable interest in fractional calculus has been stimulated by the applications that this calculus finds in numerical analysis and different areas of physics and engineering, possibly including fractal phenomena (Oldham & Spanier, 2004, Zeidler, 1995). This paper deals with the solution of the fractional integral equation of the second kind, such kind of equations appears in many problems. In particular, we have find a fractional integral equation related to many physical phenomena, such as heat flux at the boundary of a semi-infinite rod where the temperature at the boundary can be written as a fractional integral equation (Loverro ,2004). This paper is organized into three main parts. The first part begins with the proof of the existence and uniqueness of the solution of FIESK. The second part gives an analytic solution for eq. (1) based on Laplace transform. While the third part considers an approximate

solution with the aid of the collocation method to treat eq. (1). Also the stability and convergence analysis of this method are studied.

We first define a fractional integral operator I^α as follows.

Definition (1):

Let α be a nonnegative real number. For a given function $u(x)$, $x > 0$, its integral of order α is defined as follows:

$$I^\alpha u(x) = \int_0^x K(\alpha; x-t)u(t)dt$$

$$I^0 u(x) = u(x)$$

where $K(\alpha; x)$ is a monomial given by

$$K(\alpha; x-t) \equiv \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \quad x > 0, \alpha > 0$$

The fractional integral equation of the second kind has the form

$$u(x) = f(x) + \lambda I^\alpha u(x) \tag{1}$$

where $\alpha > 0$ is any fractional number, $\lambda \neq 0$ is any real number, $f(x)$ is a known function, $u(x)$ is a continuous function on $[0, b]$, i.e., $u(x) \in C[0, b]$ while I^α is the integral operator and is taken in the Riemann-Liouville (Delves & Walsh ,1997) sense which has the form as in the above definition.

1. Existence and Uniqueness of the Solution of FIESK

This section is directed toward proving the existence and uniqueness of the solution of FIESK using the following Banach fixed point theorem.

Theorem (1): Banach fixed point theorem (Zeidler,1995)

We assume that:

1. M is a closed nonempty set in the Banach space X over \mathfrak{R} ,
2. The operator $A: M \rightarrow M$ is contractive. Then the equation

$$u = Au, \quad u \in M$$

has exactly one solution u , i.e., the operator A has exactly one fixed point u on the set M .

It had been shown in (Zeidler, 1995) that the set $C[a, b]$ (with the norm $\|u(x)\|_c = \max_{a \leq x \leq b} |u(x)|$), for any two real numbers a and b such that $a < b$, is a Banach space. This fact will also be used in this section. The following lemma is needed in the proof of the existence and uniqueness results.

Lemma (1):

The Riemann-Liouville integral operator I^α , $\alpha > 0$ is a linear mapping from $C[0, b]$ into $C[0, b]$, i.e., $I^\alpha : C[0, b] \rightarrow C[0, b]$.

Proof:

First, let us prove that I^α is linear, Let $u, v \in C[0, b]$ and $\beta_1, \beta_2 \in \mathbb{R}$, then

$$\begin{aligned} I^\alpha(\beta_1 u(x) + \beta_2 v(x)) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \{ \beta_1 u(t) + \beta_2 v(t) \} dt \\ &= \frac{\beta_1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt + \frac{\beta_2}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} v(t) dt \\ &= \beta_1 I^\alpha u(x) + \beta_2 I^\alpha v(x) \end{aligned}$$

Therefore I^α is linear.

Now, let $\{u_n(x)\}_{n=1}^\infty$ be a sequence in $C[0, b]$ such that $u_n(x) \rightarrow u_0(x)$ and $u_0(x) \in C[0, b]$, we shall prove $I^\alpha u_n(x) \rightarrow I^\alpha u_0(x)$.

Since $u_n(x) \rightarrow u_0(x)$ then $\forall \varepsilon > 0, \exists k \in \mathbb{N}$ such that (Faycal. & Jacky, 2005)

$$\|u_n(x) - u_0(x)\|_c < \frac{\Gamma(\alpha+1)\varepsilon}{b^\alpha} \quad \dots(2)$$

Now, from equ.(1) we have

$$\begin{aligned} \|I^\alpha u_n(x) - I^\alpha u_0(x)\|_c &= \|I^\alpha(u_n(x) - u_0(x))\|_c \\ &= \left\| \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} (u_n(x) - u_0(x)) dt \right\|_c \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \|u_n(x) - u_0(x)\|_c dt \\ &< \frac{\Gamma(\alpha+1)\varepsilon}{b^\alpha \Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} dt \\ &= \frac{\alpha \varepsilon x^\alpha}{b^\alpha \alpha} \end{aligned}$$

Therefore, $\|I^\alpha u_n(x) - I^\alpha u_0(x)\|_c < \frac{x^\alpha \varepsilon}{b^\alpha}$

Since $x \leq b$ then

$$\|I^\alpha u_n(x) - I^\alpha u_0(x)\|_c < \varepsilon$$

That is

$$I^\alpha u_n(x) \rightarrow I^\alpha u_0(x).$$

From this we conclude that for any $u \in C[0, b]$ we get $I^\alpha u \in C[0, b]$ and this completes the proof.

Theorem (2):

Let $u \in C[0, b]$. If $|\lambda| < \frac{\Gamma(\alpha + 1)}{b^\alpha}$... (3)

then the equation (1) has a unique solution in $[0, b]$.

Proof:

For any $v \in C[0, b]$, define

$$Tv(x) = f(x) + \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} v(t) dt$$

From lemma (1) we have $I^\alpha u \in C[0, b]$, since $f \in C[0, b]$ this implies that $Tv \in C[0, b]$, i.e. $T: C[0, b] \rightarrow C[0, b]$.

To prove that T is contractive, let $v_1, v_2 \in C[0, b]$ then

$$\begin{aligned} \|Tv_1 - Tv_2\| &= \|T(v_1 - v_2)\|_c \\ &= \left\| \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} (v_1(x) - v_2(x)) dt \right\|_c \\ &\leq \frac{|\lambda|}{\Gamma(\alpha)} \|v_1 - v_2\|_c \max_{0 \leq t \leq x} \int_0^x (x-t)^{\alpha-1} dt \\ &= \frac{|\lambda| x^\alpha}{\Gamma(\alpha + 1)} \|v_1 - v_2\|_c \\ &\leq \frac{|\lambda| b^\alpha}{\Gamma(\alpha + 1)} \|v_1 - v_2\|_c \end{aligned}$$

Hence,

$$\|Tv_1 - Tv_2\|_c \leq \frac{|\lambda| b^\alpha}{\Gamma(\alpha + 1)} \|v_1 - v_2\|_c$$

Since $0 < |\lambda| < \frac{\Gamma(\alpha + 1)}{b^\alpha}$ then $0 < \frac{|\lambda| b^\alpha}{\Gamma(\alpha + 1)} < 1$, that is T is contractive.

Thus, by using theorem (1) we can conclude that T has a unique fixed point in $[0, b]$, say $u(x)$, i.e. $Tu(x) = u(x)$. This means that eq. (1) has a unique solution in $[0, b]$.

2. The Laplace Method For Fractional Integral Equation

The Laplace transform is a function commonly used in the solution of complicated equations. The formal definition of the Laplace transform is given by (Oldham & Spanier ,1974)

$$L(g(x)) = \int_0^\infty e^{-sx} g(x) dx \quad \dots(4)$$

The Laplace transform of the function $g(x)$ exists if the integral in (4) is convergent . The requirement for this is that $g(x)$ dose not grow at a rate higher than the rate at which the exponential term e^{-sx} decreases. Also, commonly used is the Laplace convolution[Oldham & Spanier,1974], given by

$$L(f(x) * g(x)) = L(f(x))L(g(x)) \quad \dots(5)$$

where $*$ is the convolution of two functions in the domain of x which is defined by

$$f(x) * g(x) = \int_0^x f(x-t)(g(t))dt$$

Other important property of Laplace transform is the Laplace transform of Riemann-Liouville integral operator of the function $g(t)$, given by (Oldham & Spanier , 1974)

$$L(I^\alpha g(x)) = s^{-\alpha} L(g(x)) \quad \dots(6)$$

Now by taking Laplace transform of both sides of eq. (1), yields

$$L(u(x)) = L(f(x)) + \lambda L(I^\alpha u(x))$$

Using eq. (6) to obtain

$$L(u(x)) = L(f(x)) + \lambda s^{-\alpha} L(u(x)) \quad \dots(7)$$

Simple rearrangements of eq. (7) gives

$$L(u(x)) = \frac{L(f(x))}{1 - \frac{\lambda}{s^\alpha}}$$

which is equivalent to

$$L(u(x)) = \left(\frac{s^\alpha}{s^\alpha - \lambda} \right) L(f(x)) \quad \dots(8)$$

The left hand side of eq. (8) can be written as

$$\left(\frac{s^\alpha}{s^\alpha - \lambda} \right) L(f(x)) = \left(s \frac{s^{\alpha-1}}{s^\alpha - \lambda} - 1 \right) L(f(x)) + L(f(x)) \quad \dots(9)$$

Substitute eq. (9) into eq. (8) to obtain

$$L(u(x)) = \left(s \frac{s^{\alpha-1}}{s^\alpha - \lambda} - 1 \right) L(f(x)) + L(f(x)) \quad \dots(10)$$

Now we want to take the inverse Laplace transform of both sides of eq. (10). In order to do this , we must address comprehend the Laplace transform of a special form of the first derivative of the Mittag-Leffler function given by (Loverro,2004)

$$L\left(\frac{d}{dx} E_\alpha(\lambda x^\alpha) \right) = s \frac{s^{\alpha-1}}{s^\alpha - \lambda} - 1 \quad \dots(11)$$

where the Mittag-Leffler function has the form

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0$$

Combining eq. (10) and eq. (11), yields

$$L(u(x)) = L\left(\frac{d}{dx} E_{\alpha}(\lambda x^{\alpha})\right)L(f(x)) + L(f(x))$$

Taking the inverse Laplace transform and using eq. (5) we get

$$u(x) = f(x) + f(x) * \frac{d}{dx} E_{\alpha}(\lambda x^{\alpha}) \quad \dots(12)$$

Eq. (12) represents a general analytic solution of eq. (1).

Example (1): Consider the following FIESK

$$u(x) = e^x(1 - \operatorname{erf}(\sqrt{x})) + I^{\frac{1}{2}}u(x), \quad 0 \leq x \leq 1$$

Since $\alpha = \frac{1}{2}$, $\lambda = 1$ and $f(x) = e^x(1 - \operatorname{erf}(\sqrt{x}))$ then the analytic solution of this problem with the aid of Laplace method is given by

$$u(x) = e^x(1 - \operatorname{erf}(\sqrt{x})) + e^x(1 - \operatorname{erf}(\sqrt{x})) \frac{d}{dx} E_{\frac{1}{2}}(\sqrt{x})$$

which is equivalent to

$$u(x) = e^x(1 - \operatorname{erf}(\sqrt{x})) + e^x(1 - \operatorname{erf}(\sqrt{x})) \sum_{k=1}^{\infty} \frac{k x^{\frac{k-1}{2}}}{\Gamma\left(\frac{k}{2} + 1\right)}$$

where $\operatorname{erf}(x)$ is the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-w^2} dw$$

3. Collocation Method For Fractional Integral Equation

In this method the solution is assumed to be a finite linear combination of some sets of analytic basis functions. However, as the number of basis functions increases we might be able to get more accurate solution to FIESK. The most important practical issue regarding such method is the choice of the basis functions $\{\varphi_i\}_{i=1}^{\infty}$. First, we set $\{\varphi_i\}_{i=1}^{\infty}$ to be a set of linearly independent elements of our space such that the span of $\{\varphi_i\}$ is dense in such space. In this paper, the following approximate solution to eq. (1) of the unknown function $u(x)$ is proposed

$$u_n(x) = \sum_{j=0}^n a_j x^j \quad \dots(13)$$

An approximate solution $u_n(x)$ will not, in general, satisfy eq. (1) exactly, and associated with such an approximate solution is the residual defined by

$$R(x;a) = u_n(x) - f(x) - \lambda I^\alpha u_n(x) \quad \dots(14)$$

By substituting eq. (13) into eq. (14), we get

$$R(x;a) = \sum_{j=0}^n a_j x^j - f(x) - \lambda I^\alpha \sum_{j=0}^n a_j x^j$$

Hence the residual function is equal to

$$R(x;a) = \sum_{j=0}^n \left(x^j - \lambda \frac{j\Gamma_\alpha}{\Gamma(\alpha + j + 1)} x^{\alpha+j} \right) a_j - f(x) \quad \dots(15)$$

The collocation method insists that the residue in eq. (15) vanishes at (n+1) collocation points $x_i \in (0,b]$; $i = 0,1,\dots,n$, this yields

$$\sum_{j=0}^n \left(x_i^j - \lambda \frac{j\Gamma_\alpha}{\Gamma(\alpha + j + 1)} x_i^{\alpha+j} \right) a_j - f(x_i) = 0, \quad i = 0,1,\dots,n \quad \dots(16)$$

Equation (16) represents a system of (n+1) equations with (n+1) unknowns a_0, a_1, \dots, a_n . Rewrite eq. (16) in matrix form as

$$Ha = B \quad \dots(17)$$

where

$$H = [h_{ij}]_{(n+1) \times (n+1)}, \quad a = (a_0, a_1, \dots, a_n)^T, \quad B = [b_i]_{(n+1)}$$

and

$$h_{ij} = x_i^j - \lambda \frac{j\Gamma_\alpha!}{\Gamma(\alpha + j + 1)} x_i^{\alpha+j}, \quad i, j = 0,1,\dots,n \quad \dots(18)$$

$$b_i = f(x_i), \quad i = 0,1,\dots,n$$

Note that x_i^0 denotes the constant 1 for any value of x_i , $i = 0,1,\dots,n$.

Finally, the system, eq.(17) can be solved by using Jacobi iterative method (Ortigueira.,2000).

4. Stability and Convergence of the Collocation Method

This section is devoted to the study of the stability and convergence of the scheme (17). The discussion is based on the fact that the Jacobi iterative method is stable and converges if the following condition holds

$$\max_{0 \leq i \leq n} \sum_{\substack{j=0 \\ j \neq i}}^n \left| \frac{h_{ij}}{h_{ii}} \right| < 1$$

Now, the main result of this section will be proved by the following theorem.

Theorem (3):

Assume that the following conditions holds:

(i) $x_i = x_0 + i\Delta x$, $i = 1, \dots, n$, where $x_0 > 0$ and Δx is the step size that must be chosen such that $x_n \leq b$.

$$(ii) \quad R(\Delta x) = \max_{0 \leq i, j \leq n} \left| \frac{1 - \frac{\lambda j \Gamma_\alpha}{\Gamma(\alpha + j + 1)} (x_0 + i\Delta x)^\alpha}{1 - \frac{\lambda i \Gamma_\alpha}{\Gamma(\alpha + i + 1)} (x_0 + i\Delta x)^\alpha} \right|$$

Then the collocation method is stable and converges to the solution of (1) if

$$\Delta x \leq \frac{1 - x_0 - R(\Delta x)}{n} \quad \dots(19)$$

Proof:

Let
$$r_i = \sum_{\substack{j=0 \\ j \neq i}}^n \left| \frac{h_{ij}}{h_{ii}} \right|$$

Since $x_i = x_0 + i\Delta x$, $i = 1, \dots, n$, using eq. (18) to obtain

$$\begin{aligned} &= \sum_{\substack{j=0 \\ j \neq i}}^n \left| \frac{1 - \frac{\lambda j \Gamma_\alpha}{\Gamma(\alpha + j + 1)} (x_0 + i\Delta x)^\alpha}{1 - \frac{\lambda i \Gamma_\alpha}{\Gamma(\alpha + i + 1)} (x_0 + i\Delta x)^\alpha} \right| \frac{(x_0 + i\Delta x)^j}{(x_0 + i\Delta x)^i} \\ r_i &= \sum_{\substack{j=0 \\ j \neq i}}^n \left| \frac{(x_0 + i\Delta x)^j - \frac{\lambda j \Gamma_\alpha}{\Gamma(\alpha + j + 1)} (x_0 + i\Delta x)^{\alpha+j}}{(x_0 + i\Delta x)^i - \frac{\lambda i \Gamma_\alpha}{\Gamma(\alpha + i + 1)} (x_0 + i\Delta x)^{\alpha+i}} \right| \\ &\leq R(\Delta x) \sum_{\substack{j=0 \\ j \neq i}}^n (x_0 + i\Delta x)^{j-i} \\ &< R(\Delta x) \sum_{j=0}^{\infty} (x_0 + n\Delta x)^j \\ &= \frac{R(\Delta x)}{1 - (x_0 + n\Delta x)} \end{aligned}$$

That is

$$r_i < \frac{R(\Delta x)}{1 - (x_0 + n\Delta x)}, \quad i = 0, 1, \dots, n$$

Therefore;

$$\max_{0 \leq i \leq n} r_i < \frac{R(\Delta x)}{1 - x_0 - n\Delta x} \quad \dots(20)$$

It is clear that if the right hand side of equ.(20) is less than or equal to one then eq. (19) will satisfied, i.e., if

$$\frac{R(\Delta x)}{1 - x_0 - n\Delta x} \leq 1$$

which leads to the required condition

$$\Delta x \leq \frac{1 - x_0 - R(\Delta x)}{n}$$

Example (2): Consider the following FIESK

$$u(x) = e^x (1 - erf(\sqrt{x})) + I^{\frac{1}{2}} u(x), \quad 0 \leq x \leq 1$$

where $erf(x)$ is the error function defined in example (1)

The exact solution of this problem is $u(x) = e^x$.

Let $n=6$, then the approximate solution takes the form

$$u_6(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$$

to find the parameters a_i ; $i=0,1,\dots,6$, let $x_i = 0.3 + 0.05 i$ for $i=0,1,\dots,6$.

The obtained results is listed in table (1)

Table (1)

x	$u(x) = e^x$	$u_6(x) = \sum_{i=0}^6 a_i x^i$
0	1	0.99995800
0.1	1.10517092	1.10492750
0.2	1.22140276	1.22055711
0.3	1.34985881	1.34800960
0.4	1.49182470	1.48857012
0.5	1.64872127	1.64365883
0.6	1.82211880	1.81484444
0.7	2.01375271	2.00385880
0.8	2.22554409	2.21261250
0.9	2.45960311	2.44321135
Least Square Error		0.00062706

5. Conclusions

Using the Jacobi iterative technique, the conditions for which the collocation method is stable and converge was provided. The present algorithm led to approximate solution which are in excellent agreement with the exact solution. The proposed method produce two sources of errors one due to the approximation and the other in the numerical treatment of the resulting system.

References

- Anatoly A. and Megumi, S.,(2002): Solution of Volterra integro-Differential Equations with Generalized Mittag-Leffler Function in the Kernels, *Journal of Integral Equation and Applicatios*, Vol.14 .
- Delves L. M. and Walsh J., (1997): Numerical Solution of Differential Equation of Fractional Order, *Electronic Transactions on Numerical Analysis*, Vol. 5, pp. 1-6.
- El-Sayed A. M. and El-Mesiry, A. E.,(2004): Numerical Solution for Multi-Term Fractional (arbitrary) Orders Differential Equations, *Mat. Apl. Comput.* Vol. 23, Petropolis.
- Faycal B. A. and Jacky, C., (2005): Fractional Differential Equations and the Schrödinger equation, *Applied Mathematics and Computation*, Vol.161, pp. 323-345.
- Irmak H. and Raina R. K.,(2004): Some Applications of Generalized Fractional Calculus Operators to a Novel Class of Analytic Functions With Negative Coefficients, *Taiwanese Journal of Mathematics*, Vol. 8, pp. 443-452.
- Loverro A.,(2004): Fractional Calculus: History, Definitions and Applications for the Engineer, A report, University of Notre Dame.
- Nagai A.,(2003): On a Certain Fractional q-Difference and its Eigen Function, *Journal of Nonlinear Mathematical Physics*, Vol. 10, supplement 2, pp. 133-142.
- Oldham K. B. and Spanier J., (1974): *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, New York and London.
- Ortigueira M. D., (2000): Introduction to Fractional Linear Systems, *IEE Proceedings vision, Image and Signal Processing*, Vol. 147 .
- Podlubny I.,(1994): *The Laplace Transform Method for Linear Differential Equations of the Fractional Order*; RNDr. Igor. Podlubny, CSc.
- Rangarajan G. and Ding M. , (2000): First Passage Time Problem for Biased Continuous-Time Random Walks, *Fractals*, Vol. 8, pp.139-145.

- Tudor C. A. and Viens F. G., Statistical Aspects of the Fractional Stochastics Calculus, e.print:pageperso. aol.fr/cipriantudor/Statfbm7.pdf.
- Wheeler N.,(1997): Construction and Physical Application of the Fractional Calculus, Notes for a Reed College Physics Seminar.
- Yuste S. B. and Acedo, L.,(2003): On an Explicit Finite Difference Method for Fractional Diffusion Equations, ar Xiv:cs.NA 10311011 Vol.10.
- Zeidler E.,Applied Functional Analysis, (1995): Applications to Mathematical Sciences, Vol. 108, Springer-Verlag, New York, Inc..

حول حل بعض المعادلات التكاملية الكسرية

سهى نجيب الراوي
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الخلاصة

قدمنا في هذا البحث المؤثر الخطي للمعادلة التكاملية الكسرية من النوع الثاني في صياغة ريمان-لوففل حيث تم التوصل الى بعض النتائج الخاصة بالوجود و الوحداية. كذلك تم التطرق الى تحويل لابلاس لمعالجة المعادلة التكاملية الكسرية من النوع الثاني، وبتطبيق هذا الأسلوب أمكن اشتقاق الحلول لأغلب المعادلات التكاملية الكسرية من النوع الثاني. الأهتمام الرئيسي الثاني في هذا البحث هو إعطاء تقريب باستخدام طريقة التجميع لحل المعادلة التكاملية الكسرية من النوع الثاني. وكذلك مناقشة اقتراب وأستقرارية الطريقة و أعطيت مقارنة بين الحلول التحليلية والتقريبية.