

## **New CG-Algorithms for the nonquadratic model**

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### **Abstract**

In this paper we have investigate a new class of conjugate gradient algorithms for unconstrained non- linear optimization which employ inexact line searches and designed for general use. Some theoretical results are investigated which ensure the local convergence of the new proposed algorithms and they compared numerically with the standard HS-CG algorithm(Hestenes & Stiefel ,1952) using a number of test functions for the dimensions between 2 to 400 with some promising numerical results .

### **Introduction**

Conjugate Gradient (CG) algorithms form a class of algorithms for minimizing a general differentiable function  $f(x), x \in R^n$ , whose gradient  $g(x)$  can be calculated, and based on the following concept of conjugacy:

If  $Q$  is a positive definite symmetric  $n \times n$  matrix then the directions  $d_1, d_2, d_3, \dots, d_n$  where  $d_k \neq 0$  and,  $k = 1, 2, 3, 4, \dots, n$ , are mutually  $Q$ -conjugate if  $d_k^T Q d_i = 0 \quad \text{for } i \neq k$

The classical algorithm in this category, proposed by Flecteher and Reeves (Fletcher & Reeves ,1964) based on the following iteration scheme:

$$x_{i+1} = x_i + \lambda_i d_i, \quad i = 1, 2, 3, \dots, m$$

where the scalar  $\lambda_i$  is the smallest positive local minimized of the one-dimensional problem

$$\min_{\lambda} f(x_i + \lambda_i d_i)$$

and  $d_i$  is a search direction generated by the equation:

$$\text{for } i = 1, 2, 3, 4, \dots, m$$

$$d_1 = -g_1$$

$$d_{i+1} = -g_{i+1} + \beta_i d_i$$

$$\beta_i = \frac{g_{i+1}^T g_{i+1}}{g_i^T g_i} \quad . \quad (\text{AL-Bayati}, 1993)$$

## **Conjugate gradient algorithm with inexact searches**

The classical CG-algorithm just discussed are well-known to be sensitive to the exactness of the line searches and this phenomenon can destroy the global efficiency of these algorithms. The higher the dimension of the problem, the bigger is the influence of this phenomenon. In order to improve the local rate of convergence and the efficiency of the classical CG-algorithm several established algorithms, namely (Nazareth ,1977; (Dixon L.C.W. ,1975; Nazareth & Nocedal ,1987) multi-step algorithms have been proposed. They have all shown that such algorithms are able to generate conjugate directions for quadratic models without performing exact searches; they will satisfy the quadratic termination property by using an error vector. Other important algorithms of this type, developed by (Sloboda ,1980; Sloboda ,1982) retain the quadratic termination property without using of an error vector.

**Corollary (1) :**Metras (Ban ,2000) proposed the following relation which it was suitable for this type of algorithms with complete proof :

$$g_{i+\frac{1}{4}} = \frac{1}{4}(g_{i+1} + 3g_i) .$$

## **General Back ground**

In this section we shall give a brief description of Sloboda algorithms (Sloboda ,1980; Sloboda ,1982) and then we discuss some theoretical properties of these algorithm.

### **1- The Sloboda algorithm (1980):**

Sloboda [Sloboda ,1980) defines a new generalized conjugate gradient algorithm for minimizing a strictly convex function of the general from  $f(x) = F(q(x))$

The outline of this algorithm is as follows :

#### **Algorithm (Sloboda ,1980)**

Step 1-Set  $x_0 \in R^n$  ,  $d_0 = -g_0$  .

Step 2-For  $i = 0,1,2,...,n-1$  Compute  $x_{i+1} = x_i + \lambda_i d_i$

where  $\lambda_i$  is chosen to satisfy the condition of the line search .

Step 3- Compute

$$w_i = \frac{d_i^T g_i}{d_i^T g_{\frac{i+1}{2}}}$$

where  $g_{\frac{i+1}{2}} = g(x_i + 1/2\lambda d_i)$  for general function  $g_{i+1} = w_i g_{\frac{i+1}{2}} - g_i$  and

Step 4-Compute the new search direction:

$$d_{i+1} = -g_{i+1} + \beta_i d_i$$

where  $\beta_i$  is defined (Hestenes & Stiefel , 1952) by :

$$\beta_i = \frac{y_i^T g_{i+1}}{y_i^T d_i} ; \quad y_i = g_{i+1} - g_i$$

Step 5- Check for convergence if  $\|g_{i+1}\| \leq \varepsilon$  ,then stop, else go to step 6.

Step 6- If  $i = n + 1$ ,then set  $i = 1$  and go to step 1. Else set  $i = i + 1$ ,and go to step 2.

## **2-The Sloboda algorithm (1982):**

Sloboda (Sloboda ,982) developed an algorithm which generates conjugate directions with imperfect searches and has the quadratic termination property without using an error vector. The outline of this algorithm for general function is as follows:

### **Algorithm (Sloboda 1982)**

Step 1-Set  $\overset{\wedge}{x}_0, \overset{\wedge}{g}_0 = g_0, \overset{\wedge}{d}_0 = -g_0$  .

Step 2-For  $i = 0,1,2,...,n$  Compute  $\overset{\wedge}{x}_{i+1} = \overset{\wedge}{x}_i + \lambda_i \overset{\wedge}{d}_i$

where  $\lambda_i$  is chosen to satisfy the condition of the line search .

Step 3- If  $\|g_{i+1}\| \leq \varepsilon$  , stop , else go to step 5 .

Step 4- If  $i = n + 1$  ,then set  $i = 1$  , and go to step 1 , else compute

$$\overset{\wedge}{g}_{i+1} = (g_{i+1} - g_i) - \frac{(g_{i+1} - g_i)^T \overset{\wedge}{g}_i}{\overset{\wedge}{g}_i^T \overset{\wedge}{g}_i} \overset{\wedge}{g}_i \quad \dots \dots \dots \quad (1)$$

Step 5- If  $\|\overset{\wedge}{g}_{i+1}\| \leq \varepsilon$  , go to step 1 , else set  $i = i + 1$  ,compute

$$\overset{\wedge}{d}_{i+1} = -\overset{\wedge}{g}_{i+1} + \frac{(g_{i+1} - g_i)^T \overset{\wedge}{g}_i}{(g_{i+1} - g_i)^T \overset{\wedge}{d}_i} \overset{\wedge}{d}_i$$

set  $i = i + 1$  and go to step 3 .

## **New CG-Algorithms with inexact line searches**

In this section, we have to present a new CG-algorithm with inexact line searches for minimizing a function  $f(x)$ .The new algorithm does not require any error vector and it depends on the vector  $g_{i+1}$ defined at the

point  $x_{i+\frac{1}{m}}$ ,  $m=1,2,3,\dots,k$ , and hence a new  $g_{i+1}^*$  which has the property of

being orthogonal to  $g_i$ . The gradient vector  $g_{i+\frac{1}{m}}$  used in this algorithm has the following property:

## New Lemma (4.1)

*Then :*

**proof:**

$$\begin{aligned}
 \frac{m}{2} g_{i+\frac{1}{m}} &= g_{i+\frac{1}{2}} + \frac{(m-2)}{2} g_i \\
 &= \frac{1}{2} (g_{i+1} + g_i) + \frac{m}{2} g_i - g_i \\
 &= \frac{1}{2} g_{i+1} + \frac{1}{2} g_i + \frac{m}{2} g_i - g_i \\
 &= \frac{1}{2} g_{i+1} + \frac{m}{2} g_i - \frac{1}{2} g_i \\
 mg_{i+\frac{1}{m}} &= g_{i+1} + mg_i - g_i \\
 \therefore g_{i+\frac{1}{m}} &= \frac{1}{m} (g_{i+1} + (m-1)g_i)
 \end{aligned}$$

Thus we have proved a new relationship to the gradient vector in Sloboda CG-algorithm.

In particular, if  $m = 4$  then we have corollary (1)

$$g_{i+\frac{1}{4}} = \frac{1}{4}(g_{i+1} + 3g_i)$$

which was defined in (Ban, 2000) as special case from our new formula.

## New (1) generalized CG-algorithm which is based on the new defined gradient vector

**Step 1-Let**  $d_0 = -g_0$ .

Step 2-For  $i = 0, 1, 2, \dots, k$  Compute  $x_{i+1} = x_i + \lambda_i d_i$

Where  $\lambda_i$  is chosen to satisfy the condition of the line search

Step 3- If  $\|g_{i+1}\| \leq \varepsilon$  , stop , else go to step 4 .

Step 4- If  $i = n + 1$ , then set  $i = 1$ , and go to step 1, else see

Step 5- If  $\|g_{i+1}^*\| \leq \varepsilon$  , set  $i = i + 1$  , go to step 2 , else set

$$\begin{aligned} d_{i+1} &= -g_{i+1}^* + \beta_i^* d_i \\ \beta_i^* &= \frac{g_{i+1}^T g_{i+1}^*}{m} \\ &= \frac{1}{m} (g_{i+1} - g_i)^T d_i \end{aligned}$$

where  $g_{i+\frac{1}{m}} - g_i = \frac{1}{m}(g_{i+1} - g_i)$   
then

$$d_{i+1} = -g_{i+1}^* + \frac{g_{i+1}^T g_{i+1}^*}{\frac{1}{m} (g_{i+1} - g_i)^T d_i} d_i \quad \dots \dots \dots \quad (5)$$

### New Lemma (4.2)

Let  $f(x)$  be a convex function and the vectors  $d_{i+1}$  and  $x_{i+1}$  are defined as in the new modified (2) CG-algorithm for the points  $x_{i+1}$  and  $x_{i+\frac{1}{m}}$ , then the following condition holds :

$$d_{i+1}^T g_{i+\frac{1}{m}} \leq 0 \quad \dots \dots \dots \quad (6)$$

**Proof :** From eq.(5) we have

$$d_{i+1} = -g_{i+1}^* + \frac{g_{i+1}^T g_{i+1}^*}{m} d_i$$

Let us multiply this direction by  $g_{i+\frac{1}{m}}^T$ , then we have :

$$g_{i+\frac{1}{m}}^T d_{i+1} = -g_{i+\frac{1}{m}}^T g_{i+1}^* + \frac{1}{(g_{i+1} - g_i)^T d_i} g_{i+\frac{1}{m}}^T d_i$$

$$= -g_{i+\frac{1}{m}}^T g_{i+1}^* + \frac{1}{(g_{i+\frac{1}{m}} - g_i)^T d_i} g_{i+\frac{1}{m}}^T d_i$$

where

$$\begin{aligned}
 \frac{1}{m}(g_{i+1} - g_i) &= g_{\frac{i+1}{m}} - g_i \\
 &= -g_{\frac{i+1}{m}}^T g_{i+1}^* \left[ 1 - \frac{g_{\frac{i+1}{m}}^T d_i}{(g_{\frac{i+1}{m}} - g_i)^T d_i} \right] \\
 &= -g_{\frac{i+1}{m}}^T g_{i+1}^* \left[ \frac{g_{\frac{i+1}{m}}^T d_i - g_i^T d_i - g_{\frac{i+1}{m}}^T d_i}{(g_{\frac{i+1}{m}} - g_i)^T d_i} \right] \\
 &= -g_{\frac{i+1}{m}}^T g_{i+1}^* \left[ -\frac{g_i^T d_i}{(g_{\frac{i+1}{m}} - g_i)^T d_i} \right]
 \end{aligned}$$

Since  $g_i^T d_i < 0$  and  $(g_{\frac{i+1}{m}} - g_i)^T d_i > 0$ , then  $-\frac{g_i^T d_i}{(g_{\frac{i+1}{m}} - g_i)^T d_i} > 0$

Now, to prove that  $-g_{\frac{i+1}{m}}^T g_{i+1}^* \leq 0$ , as follows from (4) we have

$$g_{i+1}^* = (g_{\frac{i+1}{m}} - g_i) - \frac{(g_{\frac{i+1}{m}} - g_i)^T g_i}{g_i^T g_i} g_i$$

Let us multiplying  $g_{i+1}^*$  by  $g_{\frac{i+1}{m}}^T$  then:

$$\begin{aligned}
 g_{\frac{i+1}{m}}^T g_{i+1}^* &= g_{\frac{i+1}{m}}^T (g_{\frac{i+1}{m}} - g_i) - \frac{(g_{\frac{i+1}{m}} - g_i)^T g_i}{g_i^T g_i} g_i g_{\frac{i+1}{m}}^T \\
 &= \left\| g_{\frac{i+1}{m}} \right\|^2 - g_i^T g_{\frac{i+1}{m}} - \frac{(g_{\frac{i+1}{m}}^T g_i - \|g_i\|^2)}{\|g_i\|^2} g_i^T g_{\frac{i+1}{m}} \\
 &= \frac{\|g_i\|^2 \left\| g_{\frac{i+1}{m}} \right\|^2 - \|g_i\|^2 g_i^T g_{\frac{i+1}{m}} - (g_{\frac{i+1}{m}}^T g_i - \|g_i\|^2)}{\|g_i\|^2} g_i^T g_{\frac{i+1}{m}}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\|g_i\|^2 \left\| g_{i+\frac{1}{m}} \right\|^2 - \|g_i\|^2 g_i^T g_{i+\frac{1}{m}} - g_{i+\frac{1}{m}}^T g_i g_i^T g_{i+\frac{1}{m}} - \|g_i\|^2 g_i^T g_{i+\frac{1}{m}}}{\|g_i\|^2} \\
 &= \frac{\|g_i\|^2 \left\| g_{i+\frac{1}{m}} \right\|^2 - g_{i+\frac{1}{m}}^T g_i g_i^T g_{i+\frac{1}{m}}}{\|g_i\|^2}
 \end{aligned}$$

Again

$$-g_{i+\frac{1}{m}}^T g_{i+1}^* = \frac{\|g_i\|^2 \left\| g_{i+\frac{1}{m}} \right\|^2}{\|g_i\|^2} + \frac{g_{i+\frac{1}{m}}^T g_i g_i^T g_{i+\frac{1}{m}}}{\|g_i\|^2}$$

Since

$$\begin{aligned}
 g_{i+\frac{1}{m}} &= \frac{1}{m} (g_{i+1} + (m-1)g_i), \text{ then } \left\| g_{i+\frac{1}{m}} \right\| \leq \frac{1}{m} \|g_{i+1}\| + \frac{(m-1)}{m} \|g_i\| \text{ from the Schwartz} \\
 \text{equality. This is implies that } -g_{i+\frac{1}{m}}^T g_{i+1}^* &\leq 0. \text{ Hence we get that } d_{i+1}^T g_{i+\frac{1}{m}}^* \leq 0
 \end{aligned}$$

### **The outline of the new (2) CG-algorithm:**

Step 1-Set  $x_0 \in R^n$ ,  $d_0 = -g_0$ .

Step 2-For  $i = 0, 1, 2, \dots, n-1$  and  $m = 8$  Compute  $x_{i+1} = x_i + \lambda_i d_i$

Where  $\lambda_i$  is chosen to satisfy the condition of the line search .

Step 3- Compute

$$w_i = \frac{d_i^T g_i}{d_i^T g_{i+\frac{1}{m}}}$$

Where  $g_{i+\frac{1}{m}} = \frac{1}{m} (g_{i+1} + (m-1)g_i)$  for general function  $g_{i+1}^* = w_i g_{i+\frac{1}{m}} - g_i$

Step 4-Compute the new direction:

$$d_{i+1} = -g_{i+1}^* + \beta_i d_i$$

where  $\beta_i$  is defined by :

$$\beta_i = \frac{y_i^T g_{i+1}}{y_i^T d_i}$$

Step 5- Check for convergence if  $\left\| g_{i+1} \right\| \leq \varepsilon$ , then stop, else go to step 6.

Step 6-If  $i = n$  or  $\|g_{i+1}\| \leq \varepsilon$ , then go to step 1. Else set  $i = i + 1$ , and go to step 2.

## **Numerical Results and Conclusions**

The comparison involves Five well-known test functions with twenty different versions (see appendix) with different dimension (2,4,8,10,20,40,60,80,...,400). All the results are obtained using double precision on the (Pentium (4) computer) using programs written in FORTRAN.

The compression performance of the algorithms are evaluated by considering both the total no. of function evaluations and the total no. of iterations. The stopping criterion is taken to be :

$$\|g_{i+1}\| \leq 1*10^{-5}$$

The line search routine employed is the cubic fitting technique, which uses function values and gradients.

The results are reported in Table (1) in terms of the numbers of function evaluations, the number of iterations, the results indicate that the new (2) and  $m=8$ , algorithm is more efficient. Than the standard CG-algorithm. In this a method we use the restarting criterion (Shareef, 2005) ( $d_{i+1}g_{i+1} > 0$  or  $N = K$ ).

The numerical results in Table (1) indicates that the new (2) CG-algorithm improves the standard HS-CG algorithm in about (8.98)% NOI and (7.22)% NOF respectively, for this selected test of nonlinear functions. Note that: We didn't make any numerical computations for new (1) because it was comparable with Solobodas algorithms numerically but the latter has a faster rate of convergence.

Table (1) Comparison of algorithms for  $2 \leq N \leq 400$

Test function	N	CG-algorithm NOI (NOF)	new(2),m=8 NOI (NOF)
DIXON	2	32 (67)	6 (18)
SHALLO	2	48 (101)	7 (25)
ROSEN	4	27 (72)	27 (72)
NON-DIAGON	4	23 (65)	23 (57)
WOOD	40	48 (101)	45 (95)
WOLFE	40	47 (95)	47 (95)
ROSEN	60	22 (55)	21 (54)
CUBIC	60	11 (32)	11 (32)
POWELL	80	84 (193)	85 (202)
WOLFE	80	49 (99)	49 (99)
NON-DIAGON	100	25 (62)	22 (55)
WOOD	100	85 (175)	74 (152)
POWELL	100	113 (264)	114 (275)
SHALLO	200	6 (17)	6 (19)
WOOD	200	85 (175)	74 (152)
WOLFE	200	51 (103)	51 (103)
POWELL	400	415 (871)	402 (860)
WOOD	400	86 (176)	75 (154)
CUBIC	400	12 (35)	12 (35)
WOLFE	400	54 (109)	54 (109)
TOTAL		1323 (2867)	1205 (2663)

Percentage performance of new(2) algorithm against the standard CG-algorithm

Tools	Standard –CG	NEW
NOI	100	91.02
NOF	100	92.88

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## **Appendix**

1. *Cubic function* :

$$f(x) = \sum_{i=1}^{n/2} (100(x_{2i} - x_{2i-1}^3)^2 + (1 - x_{2i-1})^2)$$

*Starting point:*  $(-1.2, 1, -1.2, 1, \dots)^T$

2. *Non-diagonal function* :

$$f(x) = \sum_{i=1}^{n/2} (100(x_i - x_i^3)^2 + (1 - x_i)^2)$$

*Starting point:*  $(-1, \dots)^T$

3. *Rosenbrock function* :

$$f(x) = \sum_{i=1}^{n/2} (100(x_{2i} - x_{2i-1}^3)^2 + (1 - x_{2i-1})^2)$$

*Starting point:*  $(-1.2, 1, -1.2, 1, \dots)^T$

4. *Generalized Powell function* :

$$f(x) = \sum_{i=1}^{n/4} (x_{4i-3} - 10x_{4i-2})^2 + 5(x_{4i-1} - x_{4i})^2 + (x_{4i-1} - 2x_{4i})^2 + 10(x_{4i-9} - x_{4i})^4 + (x_{4i-2} - 2x_{4i-1} - x_{4i})^2$$

*Starting point:*  $(3, 1, 0, 1, \dots)^T$

5. *Shallow function* :

$$f(x) = \sum_{i=1}^{n/2} (x_{2i-1}^2 - x_{2i})^2 + (1 - x_{2i-1})^2$$

*Starting point:*  $(-2, -2, \dots)^T$

6. *Dixon function* :

$$f(x) = (1 - x_1)^2 + (1 - x_0)^2 + \sum_{i=2}^9 (x_i^2 - x_{i-1})^2$$

*Starting point:*  $(-1, \dots)^T$

7. *Welfe function* :

$$f(x) = (-x_1(3 - x_1/2) + 2x_2 - 1)^2 + \sum_{i=1}^{n-1} (x_{i-1} - x_i(3 - x_i(3 - x_i/2) + 2x_{i+1} - 1)^2 + (x_{n+1} - x_n(3x_n/2 - 1))^2$$

*Starting point:*  $(-1, \dots)^T$

8. *Wood function* :

$$f(x) = \sum_{i=1}^{n/4} 100(x_{4i-2} + x_{4i-3}^2)^2 + (1 - x_{4i-3})^2 + 90(x_{4i} - x_{4i-1}^2)^2 + (1 - x_{4i-1}^2)^2 + 1.0$$

*Starting point :*  $(-3, -1, -3, -1, \dots)^T$

## خوارزميات جديدة للتدرج المترافق في نموذج غير تربيعي

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### الخلاصة

في هذا البحث تم التطرق إلى صنف جديد من خوارزميات الاتجاهات المترافقة في الأمثلة اللاحظية باستعمال خط بحث غير تام وصممت للاستخدام العام . أعطيت بعض النتائج النظرية التي تؤكد التقارب الموضعي للخوارزميات الجديدة المقترحة . كما تمت مقارنة هذه الخوارزميات مع خوارزمية (Hestenes & Stiefel) باستخدام عدد من الدوال الاختبارية وبأبعاد تتراوح من ٢ إلى ٤٠٠ مع الحصول على نتائج مشجعة.