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A direct Approximation Method to solve OCP Using Laguerre Functions

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Abstract

This paper presents an approximate method to solve unconstrained optimal control problem (OCP).This method is classified as a direct method in which an OCP is converted into a mathematical programming problem.The proposed direct method is employed by using the state parameterization technique with the aid of Laguerre polynomials and Laguerre functions to approximate the system state variables. To facilitate the computations within this method, new properties their proofs of Laguerre polynomials and Laguerre functions are given with proof.Furthermore, we will derive the condition under which the proposed method with Laguerre functions converges to the solution of the OCP equation. We will also show that for N (the number of basis functions) sufficiently large, the approximate states stabilize the system.The proposed method has been applied on several numerical examples and we find that it gives better or comparable results compared with some other methods.

Introduction

Optimal control is a special type of the optimization problem and has tremendous applications. It deals with the study of systems. The early developments of the theories were given by engineers whose systems were machines and their interactions were controls. However, nowadays the system covers a much wider range, such as the human body or a particular system of the human body, a particular industry or even the whole economy of a country. (Alonzo & Bryan ,2002, Beeler & Tran,1999, Herdman & Morin, 2002, Jankowski,2002).

In order to understand a system, a mathematical equation representing it exactly or to a reasonable approximation is written. Usually a system is represented by one of the equation: differential, partial-differential, integral, integro-differential, difference, stochastic-differential, stochastic-integral, and such equations are known as models of the system. The objective of optimal or an optimal $u^*(t)$ control is to determine an optimal open loop control that forces the system to satisfy physical constraints $u^*(x,t)$ feedback control and at the same time minimizes or maximizes a performance index. (Auzinger & Kneisl ,2002 ,Findeisen & Diehl, 2001).Systems governed by ordinary differential equations arise in many applications as, e.g. , in astronautics, aeronautics, robotics, and economics . The task of optimizing these systems leads to the OCP investigated in this paper.The general OCP formulation is to find a control vector $u(t)$ that minimize the functional (Hussein, 1998)

$$J(x(t),u(t)) = \Phi(x(t_f)) + \int_{t_0}^{t_f} F(t,x(t),u(t))dt \qquad \ldots(1)$$

subject to a system of n differential equations

$$\dot{x}_i(t) = f_i(x(t),u(t),t), \quad i = 1,2,\ldots n \quad 0 \le t \le t_f \qquad \ldots(2)$$

with the boundary conditions $r_i(x(0),x(t_f),t_f) = 0$, $i = 1,2,\ldots,k \le 2n$

Here, the l vector of control variables is denoted by $u(t) = (u_1(t),\ldots,u_l(t))^T$ and the n vector of state variables is denoted by $x(t) = (x_1(t),\ldots,x_n(t))^T$.The functions $\Phi : \Re^{n+1} \to \Re$, $f : \Re^{n+l+1} \to \Re^n$, are assumed to be continuously differentiable. The controls $u_i : [0,t_f] \to \Re$, $i = 1,2,\ldots l$ are assumed to be bounded and measurable and t_f may be finite or infinite.

The work throughout this work is concerned with the QOC problems and is associated with both finite and infinite time of minimizing a running cost or performance index subject to linear control dynamics.(Binder & Blank, 2001). The mathematical formulation of the problems addressed in this paper is given by:

P1*:* Finite time horizon linear quadratic optimal control FLQOC problem, where the optimization index (performance index) , eqn. (1) , is over a finite time interval

$$J = \int_{t_0}^{t_f} (x^T Q x + u^T R u)dt \qquad \ldots(3)$$

subject to the linear system state equations

$$\dot{x} = Ax + Bu \qquad \ldots(4)$$

satisfying the initial conditions

$$x(t_0) = x_0 \qquad \ldots(5)$$

where $A \in \Re^n \times \Re^n$, $B \in \Re^n \times \Re^l$, $x \in \Re^n$, $u \in \Re^l$, Q is $n \times n$ positive semi definite matrix , $x^T Q x \ge 0,$ and R is a $l \times l$ positive definite matrix, $u^T R u > 0$ unless $u(t) = 0$.

P2: Infinite time horizon linear quadratic optimal control ILQOC problem, where the optimization index , eqn. (3), is over an infinite time interval.

The Basic Idea of the Method

The direct methods can be applied by using either the discretization or the parameterization techniques for : control variables, both state and control variables or state variables. Throughout this work, state parameterization technique is used. In this technique, Laguerre polynomials and Laguerre functions will be used to approximate the solution to the finite and infinite OCP respectively. (Marta & Werner 1995, Oskar & Bulirsch,1992).

To apply state vector parameterization, it is first necessary to place the solution to the differential equation in a Hilbert space. To do so we restrict attention to a compact subset Ω of the stability region of a known stabilizing control. When the solutions to the OCP equation are restricted to this set, they exist in the Hilbert space $L_2(D)$.

To use the Laguerre polynomials for state vector parameterization of finite time horizon OCP, eqn. (3), on the time interval $t \in [0, t_f]$, any time function $s(t)$, $0 \le t \le t_f$, can be approximated by adding a number of Laguerre polynomials as follows,

$$s(t) \approx \sum_{i=0}^{N-1} a_i L_i(t) \qquad \dots (6)$$

So eqn. (6) is a parameterization of the function $s(t)$, $0 \le t \le t_f$, by Laguerre polynomials and $a_0, a_1, \dots, a_{N-1}$ are the N parameters called Laguerre coefficients. Therefore,

$$x_N(t) = \sum_{i=0}^{N-1} a_i L_i(t) \qquad \dots (7)$$

is called a Laguerre series approximation of $s(t)$, $0 \le t \le t_f$.

The following three properties of Laguerre series will be especially useful to solve OCP by means of state parameterization,

- The recurrence relation

$$\dot{L}_n(t) = n\dot{L}_{n-1}(t) - nL_{n-1}(t) \qquad \dots (8)$$

- The initial values

$$L_n(0) = n! \qquad \dots (9)$$

- Integration property

$$\int_0^x L_n(t)dt = L_n(x) - \frac{L_{n+1}(x)}{n+1} \qquad \dots (10)$$

Moreover; an important new formula of Laguerre polynomials is derived in this paper.

Lemma (1):

The first derivative of Laguerre polynomials $L_n(t)$, is formulated as:

$$\dot{L}_n(t) = -n!\sum_{i=0}^{n-1}\frac{L_i(t)}{i!} \qquad n \geq 1 \qquad \dots (11)$$

Proof: (Hussein ,1998**)**
The mathematical induction principle is used to prove this lemma.
eqn. (11) is true for $n=0$ and $n=1$ by direct calculation,
Since $L_0(t)=1$ and $L_1(t)=-t+1$, we have $\dot{L}_0(t)=0$ and $\dot{L}_1(t)=-L_0$.
Let us assume that eqn. (11) is true for a particular positive integer $n=k$, i.e.,

$$\dot{L}_k(t) = -k!\sum_{i=0}^{k-1}\frac{L_i(t)}{i!} \qquad \dots (12)$$

Now, we wish to see that eqn. (12) is true for $n=k+1$. Using the recurrence relation (8) with $n=k+1$,

$$\begin{aligned}\dot{L}_{k+1} = (k+1)\dot{L}_k(t) - (k+1)L_k(t) &= (k+1)\left[\dot{L}_k(t) - L_k(t)\right] \\ &= (k+1)\left[-\sum_{i=0}^{k-1}\frac{k!}{i!}L_i(t) - L_k(t)\right] \\ &= -\sum_{i-0}^{k-1}\frac{(k+1)!}{i!}L_i(t) - (k+1)L_k(t) \\ &= -\sum_{i=0}^{k}\frac{(k+1)!}{i!}L_i(t)\end{aligned}$$

Hence $\qquad \dot{L}_{k+1}(t) = \sum_{i=0}^{k}\frac{(k+1)!}{i!}L_i(t)$

Thus, since eqn. (11) is true for $n=0$, so it is valid for $n=1,2,\dots$.

<u>Remark</u>

This lemma is useful for our computational work because it relates Laguerre series approximations of time functions to Laguerre series approximations of the time derivatives of these time functions. The idea of the state vector parameterization, using the Laguerre polynomials L_i, as a basis functions, is to approximate the state variables as follows:

Approximate the state variables by a finite length Laguerre series, i.e.

$$x_j(t) \approx x_j^N(t) = \sum_{i=0}^{N} a_{ij} L_i(t) \ , \quad j = 1,2,\dots n \qquad \dots (13)$$

where a_{ij}'s are the unknown parameters and the $L_i(t)$'s are functions in $L_2(a,b)$.

The control variables $u_k(t)$, $k = 1,2,\dots m$, are determined from the system state equations as a function of the unknown parameters of the state variables. Note that, two cases can be distinguished when applying SVP technique.

Case (1):

If the numbers of the states and the control variables are equal $n = m$, then each state variable will be approximated by a finite length Laguerre series and the control vector is obtained as a function of the state variables (13). Therefore, in this case, the matrix B in eqn. (4) is assumed to be nonsingular, then the control variables can be obtained as:

$$u_k(t) = B^{-1}\left(\dot{x}_j(t) - Ax_j(t)\right) \ ; \quad j,k = 1,2,\dots n = m \qquad \dots (14)$$

Then by substituting eqn. (13) into eqn. (14), yields:

$$u_k(t) = B^{-1}\left(\sum_{i=0}^{N} a_{ij}\dot{L}_i(t) - A\sum_{i=0}^{N} a_{ij}L_i(t)\right) \qquad \dots (15)$$

Rewrite $\dot{L}_i(t)$ in terms of $L_i(t)$, then reordering eqn. (15) by collecting the coefficients of the same $L_i(t)$, yields:

$$u_k(t) = \sum_{i=0}^{N} b_{ik}\phi_i(t) \ , \qquad k = 1,2,\dots,n = m \qquad \dots (16)$$

where b_i's are expressed in terms of a_i's.

Case (2):

If the number of the state variables is greater than the number of control variables, i.e., $n > m$, then there is no need to approximate all the state variables. In this case, a set of the state variables are approximated which will enable us to find the remaining state variables and control variables as a functions of this set, so that the quadratic optimal control problem (3)-(5) is reduced to a quadratic programming problem with fewer unknown parameters. Then the initial conditions (5) are replaced by equating constraints as follows:

Since $$x_j(t)=\sum_{i=0}^{N} a_{ij} L_i(t)\,, \qquad j=1,2,\ldots n$$

Hence, $$x_j(0)=\sum_{i=0}^{N} a_{ij} L_i(0)$$

or $$\sum_{i=0}^{N} a_{ij} L_i(0)-x_j(0)=0\,, \qquad j=1,2,\ldots n \qquad \ldots (17)$$

Eqn. (17) represents the equality constraints.

By substituting the approximations (13) and (16) of the state and control variables respectively into the performance index (3), we can get the approximate performance index value J^*, so that the LQOC problem (3)-(5) can be converted into a quadratic function of the unknown parameters a_{ij} as follows:

Rewrite eqns. (13) and (16) in the forms

$$x=\alpha L \quad \text{and} \quad u=\beta L$$

where $$\alpha=\begin{pmatrix} a_{01} & a_{11} & \cdots & a_{N1} \\ a_{02} & a_{12} & \cdots & a_{N2} \\ \vdots & \vdots & & \vdots \\ a_{0n} & a_{1n} & \cdots & a_{Nn} \end{pmatrix}, \qquad \beta=\begin{pmatrix} b_{01} & b_{11} & \cdots & b_{N1} \\ b_{02} & b_{12} & \cdots & b_{N2} \\ \vdots & \vdots & & \vdots \\ b_{0m} & b_{1m} & \cdots & b_{Nm} \end{pmatrix}$$

and $L=(L_0 \quad L_1 \quad \cdots \quad L_N)^T$.

Therefore, the formula of approximate performance index J^* is equal to

$$J^*=\int_0^{t_f}(\phi^T\alpha^T Q\alpha\phi+\phi^T\beta^T R\beta\phi)dt$$

Let $V=\alpha^T Q\alpha$ and $W=\beta^T R\beta$, yields:

$$J^*=\int_0^{t_f}(\phi^T V\phi+\phi^T W\phi)dt \qquad \ldots (18)$$

The first term of the integrand in (18) can be written as:

$$\begin{aligned} L^T VL = v_{11}L_0L_0 &+ 2v_{12}L_0L_1 + 2v_{13}L_0L_2 + \cdots + 2v_{1,N+1}L_0L_N \\ &+ v_{22}L_1L_1 + 2v_{23}L_1L_2 + \cdots + 2v_{2,N+1}L_1L_N \\ &\quad + v_{33}L_2L_2 + \cdots + 2v_{3,N+1}L_2L_N \\ &\qquad \ddots \qquad\qquad \vdots \\ &\qquad\qquad + v_{N+1,N+1}L_NL_N \end{aligned} \qquad \ldots (19)$$

Also the second term of the integrand in (18) can be written in the same way. Now, the QOC problem (3)-(5) is converted into parameters optimization problem which is quadratic in the unknown parameters and the new problem can be stated as:

$$\min_a \quad J^*=\frac{1}{2}a^T Ha \qquad \ldots (20)$$

Subject to

$$Fa - b = 0 \qquad \ldots(21)$$

The matrix H can be defined by finding the Hessian of J^*,

$$H = \frac{\partial^2 J^*}{\partial a_{ik} \partial a_{jk}}, \qquad i, j = 0,1,2,\ldots N \qquad \ldots(22)$$

where $k = 1,2,\ldots n$ if $n = m$ and $k = 1,2,\ldots,q < n$ if $n > m$.
The constraints (21) are due to the initial conditions (17).

Remark

The state equations which are not satisfied yet are added to the initial conditions to represent equality constraints.
Finally, the optimal value of the vector a^* can be obtained from the standard quadratic programming method discussed

$$a^* = H^{-1} F^T (F H^{-1} F^T)^{-1} b$$

Now to solve the following infinite LQOC Problems using Laguerre functions,

Minimize $$J = \int_0^\infty (x^T Q x + u^T R u) dt \qquad \ldots(23)$$

subject to $$\dot{x} = Ax + Bu \qquad \ldots(24)$$

$$x(0) = x_0 \qquad \ldots(25)$$

An important new formula of Laguerre polynomials is derived in this paper.

Lemma (2):

The first derivatives of Laguerre functions is given by:

$$\dot{l}_n(t) = -\frac{1}{2} l_n(t) - \sum_{i=1}^{n} \frac{n!}{(n-i)!} l_{n-i}(t)$$

Proof:

Since the nth Laguerre functions $l_n(t)$ is defined by

$$l_n(t) = e^{-t/2} L_n(t)$$

Then differentiating Laguerre functions with respect to t, yields

$$\dot{l}_n(t) = e^{-t/2} \dot{L}_n(t) - \frac{1}{2} e^{-t/2} L_n(t) = e^{-t/2} \left[\dot{L}_n(t) - \frac{1}{2} L_n(t) \right]$$

with the aid of lemma (1) , we get

$$\dot{l}_n(t) = e^{-t/2} \left[-n! \sum_{i-0}^{n-1} \frac{L_i(t)}{i!} - \frac{1}{2} L_n(t) \right] = -\frac{1}{2} e^{-t/2} L_n(t) - n! e^{-t/2} \sum_{i=0}^{n-1} \frac{L_i(t)}{i!}$$

since $\sum_{i=0}^{n-1}\frac{L_i(t)}{i!}$ is equal to $\sum_{i=1}^{n}\frac{L_{n-i}(t)}{(n-i)!}$

Then $\dot{l}_n(t)=-\frac{1}{2}l_n(t)-\sum_{i=1}^{n}\frac{n!}{(n-i)!}\,l_{n-i}(t)$ which is the required result.

The algorithm represents the state vector parameterization with the use of Laguerre functions which can be summarized as follows:

- Approximate the state variables by using a finite number of basis Laguerre functions, such that:

If $n=m \;\Rightarrow\; x_j(t)\approx x_j^N(t)=\sum_{i=0}^{N}a_{ij}l_i(t),\quad j=1,2,\ldots,n$

If $n>m \;\Rightarrow\; x_j(t)\approx x_j^N(t)=\sum_{i=0}^{N}a_{ij}l_i(t),\quad j=1,2,\ldots,q<n$

- Determine the control variables $u_k(t)$, $k=1,2,\ldots,m$, such that:

If $n=m$ then use (16) to obtain $u_k(t)$

$$u_k(t)=B^{-1}\left(\sum_{i=0}^{N}a_{ij}\dot{l}_i(t)-A\sum_{i=0}^{N}a_{ij}l_i(t)\right)$$

with the aid of the differentiation formula for Laguerre functions which was given through lemma (2), we obtain

$$u_k(t)=B^{-1}\left[\sum_{i=0}^{N}a_{ij}\left(-\frac{1}{2}l_i(t)-\sum_{j=1}^{i}\frac{i!}{(i-j)!}l_{i-j}(t)\right)-A\sum_{i=0}^{N}a_{ij}l_i(t)\right]$$

which can be rewritten as:

$$u_k(t)=\sum_{i=0}^{N}b_{ik}l_i(t),\qquad k=1,2,\ldots,n=m \qquad \ldots(26)$$

where b_{ik}'s are expressed in terms of a_{ij}'s.

If $n>m$, then

$$u_k(t)=\sum_{i=0}^{N}b_{ik}l_i(t),\quad k=1,2,\ldots m$$

- Evaluate the two matrices $V=\alpha^TQ\alpha$ and $W=\beta^TR\beta$, then test for illcondition , that is we compute the condition number of V and W ,

$cond(V)=\|V\|_2\,\|V\|_2^{-1}$ and $cond(W)=\|W\|_2\,\|W\|_2^{-1}$

- Determine the two terms l^TVl and l^TWl, then consider the integral from $t=0$ to $t=\infty$, such that:

Since Laguerre functions are orthogonal over $[0,\infty)$, i.e.,

$$\int_0^{\infty}l_n(t)l_m(t)dt=\begin{cases}0 & n\neq m\\ (n!)^2 & n=m\end{cases}$$

Hence $\int_0^\infty l^T V l dt = v_{11} + v_{22} + (2!)^2 v_{33} + (3!)^2 v_{44} + \cdots + (N!)^2 v_{N+1,N+1} = \sum_{i=0}^{N} (i!)^2 v_{i+1,i+1}$

similarly, $\int_0^\infty l^T W l dt = \sum_{i=0}^{N} (i!)^2 w_{i+1,i+1}$

- Find an expression of J^*, such that

$$J^* = \int_0^\infty (l^T V l + l^T W l) dt = \sum_{i=0}^{N} (i!)^2 \left[V_{i+1,i+1} + W_{i+1,i+1} \right] \qquad \ldots(27)$$

- Compute the Hessian for J^* using (22).
- Use the initial values of Laguerre functions to find the equality constraints

$$\sum_{i=0}^{N} a_{ij} l_i(0) - x_j(0) = 0\,, \qquad j = 1,2,\ldots n$$

since $l_i(0) = i!$, therefore, the equality constraints become

$$a_{0j} + a_{1j} + 2! a_{2j} + 3! a_{3j} + \cdots + N! a_{Nj} - x_j(0) = 0\,, \quad j = 1,2,\ldots n \qquad \ldots(28)$$

Eqn. (28) can be rewritten in the form (21) and thus we can find the matrices F and b .

- Use the standard quadratic programming method to find the optimal parameters a_{ij}; $i = 0,1,2,\ldots,N$; $j = 1,2,\ldots,q \le n$.
- Find the optimal value J^* using eqn. (27).

The Convergence Test

In the proposed method, the state vector will be approximated globally, over the entire domain of the problem. To do so, we assumes a global functional form for the solution, typically an expansion in terms of a set of orthogonal functions (basis set) or at least linearly independent set,

$$x_i(t) = \sum_{k=1}^{\infty} a_{ik} \phi_k(t) \qquad i = 1,2,\ldots,n \qquad \ldots(29)$$

It is not possible to perform computations on an infinite number of terms, therefore; we must truncate the series in eqn. (29). In place of (29), we take

$$x_{iN}(t) = \sum_{k=1}^{N} a_{ik} \phi_k(t)$$

so that $$x_i(t) = x_{iN}(t) + \sum_{k=N+1}^{\infty} a_{ik} \phi_k(t) = x_{iN}(t) + r_i(t) \qquad \ldots(30)$$

we must select coefficients in eqn. (30) such that the norm of the residual function $\|r(t)\|$ is less than some convergence criterion ε , where

$$r(t) = \max(r_1(t), r_2(t), \ldots, r_N(t))$$

Now we will return to the question of how large N must be later. An important of SVP method is that the residual function decreases very rapidly. In fact, SVP in general are of infinite order, that is, the norm of the residual $\|r(t)\|$ approaches zero faster than any finite power of $(1/N)$.

There is a convergence test that must be used with SVP method. It is to do with the number of terms kept in the basis set N. The most useful test of convergence in terms of N comes from examining the L^2 norm of x_i, $i=1,2,\ldots,n$ (the state variables that is approximated using Laguerre functions), i.e.,

$$\left[\int_0^\infty (x_i(t)-x_{iN}(t))^2\,dt\right]^{\frac{1}{2}} < \varepsilon_i, \qquad i=1,2,\ldots,n$$

Let $\varepsilon = \max(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)$, therefore

$$\left[\int_0^\infty (x(t)-x_N(t))^2\,dt\right]^{\frac{1}{2}} < \varepsilon$$

for all N greater than some value N_0. Since we do not know $x(t)$, we replace the presumably better approximation $x_{N+M}(t)$, where $M \geq 1$

$$\left[\int_0^\infty (x_{N+M}(t)-x_N(t))^2\,dt\right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow \quad \left[\int_0^\infty \left(\sum_{i=0}^{N+M} a_i l_i(t) - \sum_{i=0}^{N} a_i l_i(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow \quad \left[\int_0^\infty \left(\sum_{i=N+1}^{N+M} a_i l_i(t)\right)^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow \quad \left[\int_0^\infty \left(\sum_{i=N+1}^{N+M} a_i l_i(t)\right)\left(\sum_{i=N+1}^{N+M} a_i l_i(t)\right) dt\right]^{\frac{1}{2}} < \varepsilon$$

$$\Rightarrow \quad \sum_{i=N+1}^{N+M}\sum_{j=N+1}^{N+M} a_i a_j \int_0^\infty l_i(t)\, l_j(t)\, dt < \varepsilon \qquad \ldots(31)$$

Since Laguerre functions $l_i(t)$ are orthogonal functions over $[0,\infty)$, it can be shown that (31) reduces to the simple form

$$\sum_{k=N+1}^{N+M} (k!)^2\, a_k^{\;2} < \varepsilon$$

In other words, when the sum of the squares of the remaining coefficients becomes negligible, then we have a satisfactory approximation to the solution.

Numerical Examples

Example (1):

The following finite linear quadratic problem is considered ,

Minimize

$$J = \int_0^1 \left(x_1^{\ 2} + x_2^{\ 2} + 0.005u^2\right)dt \qquad \ldots(32)$$

Subject to

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0 \qquad \ldots(33)$$

$$\dot{x}_2 = -x_2 + u \qquad x_2(0) = -1 \qquad \ldots(34)$$

The optimal value of the performance in this problem is 0.06936094. This example contains two state variables $x_1(t)$ and $x_2(t)$ and one control variable $u(t)$, i.e., $n = 2$ and $m = 1$.

Here $x_1(t)$ is approximated by $5th$ order Lagueerre series of unknown parameters, then $x_2(t)$ is found from (33) using the differentiation property of the Laguerre polynomials that is used. The control variable $u(t)$ is obtained from (34). By substituting $x_1(t)$, $x_2(t)$ and $u(t)$ into (32), an expression of J^* can be found. In this approach, the state variables $x_1(t)$ and $x_2(t)$ as well as the control variable $u(t)$ are approximated to be:

$$x_1(t) \approx \sum_{i=0}^{5} a_i L_i(t)$$

$$\begin{aligned} x_2(t) \approx -[&(a_1 + 2a_2 + 6a_3 + 24a_4 + 120a_5)L_0(t) \\ &+ (2a_2 + 6a_3 + 24a_4 + 120a_5)L_1(t) \\ &+ (3a_3 + 12a_4 + 60a_5)L_2(t) \\ &+ (4a_4 + 20a_5)L_3(t) \\ &+ 5a_5 L_4(t)] \end{aligned}$$

and

$$\begin{aligned} u(t) = [&(6a_3 + 48a_4 + 360a_5 - a1)L_0(t) + (24a_4 + 240a_5 - 2a_2)L_1(t) \\ &+ (60a_5 - 3a_3)L_2(t) - 4a_4 L_3(t) - 5a_5 L_4(t)] \end{aligned}$$

Then the expression for J^* will be equal to:

$$
\begin{aligned}
J^* = a_0^{\,2} + a_0a_1 + \frac{2}{3}a_0a_2 - \frac{1}{2}a_0a_3 - \frac{38}{5}a_0a_4 - \frac{151}{3}a_0a_5 \\
+ \frac{803}{600}a_1^{\,2} + \frac{2053}{300}a_1a_2 - \frac{427}{20}a_1a_3 - \frac{23957}{300}a_1a_4 + \frac{731701}{2100}a_1a_5 \\
+ \frac{1531}{150}a_2^{\,2} + \frac{41899}{600}a_2a_3 + \frac{744017}{2625}a_2a_4 + \frac{706891}{525}a_2a_5 \\
+ \frac{892513}{7000}a_3^{\,2} + \frac{3855827}{3500}a_3a_4 - \frac{3515917}{6300}a_3a_5 \\
+ \frac{567511}{225}a_4^{\,2} + \frac{9731551}{360}a_4a_5 \\
+ \frac{422309369}{5544}a_5^{\,2}
\end{aligned}
$$

Note that, for simplification, a_i is written instead of a_{i1}, $i = 0,1,\ldots,5$.
The equality constraints are:

$$a_0 + a_1 + 2a_2 + 6a_3 + 24a_4 + 120a_5 = 0$$

$$-a_1 - 4a_2 - 18a_3 - 96a_4 - 600a_5 + 1 = 0$$

Comparing the above constraints with the equation $Fa - b = 0$, we have

$$F = \begin{pmatrix} 1 & 1 & 2 & 6 & 24 & 120 \\ 0 & -1 & -4 & -18 & -96 & -600 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

while $\quad a = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix}^T$

When using quadratic programming method, the optimal parameters are obtained :

$$a = \begin{pmatrix} -19.01157449 & 106.96044577 & -120.63662261 & 45.45612945 & -64.39072014 & 2.92711047 \end{pmatrix}^T$$

The optimal trajectories and the optimal control are:

$$x_1(t) = \begin{pmatrix} -19.01157449 & 106.96044577 & -120.63662261 & 45.45612945 \\ -64.39072014 & 2.92711047 \end{pmatrix}^T L(t)$$

$$x_2(t) = \begin{pmatrix} 541.6900279 & 434.7295833 & 460.6936251 & 199.0206712 & -14.6355537 & 0 \end{pmatrix}^T L(t)$$

$$u(t) = \begin{pmatrix} -601.5975254 & -1871.218467 & 39.25823985 & 257.5628806 & -14.6355537 & 0 \end{pmatrix}^T L(t)$$

while the optimal value is $J^* = 0.07595220886$.

Also this problem is solved by expanding $x_1(t)$ into different orders of Laguerre series. Table (1) shows the values of the optimal parameters with the optimal value for $N = 7$ of Laguerre series .

Table (1)
Laguerre Parameters of order N=7

i	Laguerre Parameters		
	a_{i1}	a_{i2}	b_{i1}
0	-21469.28568404	-21469.28568404	-42937.57136873
1	163312.09351802	141842.80783398	262217.32998393
2	-266391.73172890	-95470.32781191	-330753.39454885
3	161057.47851417	95900.70257687	146707.04957476
4	-43852.14607425	-19876.97043003	-27052.35411059
5	5735.84131792	1760.44723191	2085.81772771
6	-347.63628795	-54.22841597	-54.22841597
7	7.74691657	0	0
J	0.06969730		
$\lvert J_{exact} - J_{app.} \rvert$	3.3636×10^{-4}		

This example was solved by (Hsieh,1965) using a modified steepest method and by (Neuman &Sen,1973) using collocation and approximation by cubic splines, while (Hussian ,1998) treated this example by using state parameterization with Chebyshev polynomials. These results besides the present results are listed in Table (2).

Table (2)
Optimal values of J for example (1)

Source		J	$\lvert J_{exact} - J_{app.} \rvert$
Exact value		0.06936094	_
Hsieh [7]		0.0702	8.3906×10^{-4}
Neuman&Sen[13]	N=4	0.0703	9.3906×10^{-4}
	N=9	0.06989	5.2906×10^{-4}
Hussian [8]	N=5	0.07595646	6.5955×10^{-3}
	N=9	0.0693689	7.96×10^{-6}
Our Research	N=4	0.09172786	2.2367×10^{-2}
	N=5	0.07595221	6.5913×10^{-3}
	N=6	0.07098744	1.6265×10^{-3}
	N=7	0.06969730	3.3636×10^{-4}
	N=8	0.06941721	5.627×10^{-5}
	N=9	0.06936891	7.97×10^{-6}

Example (2):

The algorithm using Laguerre functions is tested on the following infinite nonlinear QOC problem.

Minimize

$$J=\int_0^{\infty}\left\{x^T\begin{pmatrix}0.5 & 0\\ 0 & 0.5\end{pmatrix}x+0.5u^2\right\}dt$$

subject to

$$\dot{x}_1 = x_2 \qquad\qquad x_1(0)=1$$

$$\dot{x}_2 = x_1^3 + u \qquad\qquad x_2(0)=0$$

The linearized system about $(x_1,x_2)=(0,0)$ for this problem is treated using Laguerre functions. The parameters a_{1i} , a_{2i} and b_{1i} ; $i=0,1,\ldots,5$ are listed in table(3).

Table (3)
The Parameters of Laguerre Functions for N=5

i	*SVPl* $(N=5)$		
	a_{i1}	a_{i2}	b_{i1}
0	1.05485473	−0.47257263	−0.23628632
1	0.16538476	0.13754711	−0.40379908
2	−0.03721944	0.09151002	−0.12175775
3	−0.01387732	0.01736144	−0.01665352
4	−0.01579266	0.00164802	−0.00116919
5	−0.00013807	0.00006904	−0.00003452
J	0.86607560		

In Table (4) a comparison between the computed optimal value obtained by using the algorithm with Laguerre functions for different orders.

Table (4)
Minimum Values of for N=2,3,4,5,6

N	*SVPl*
2	0.96292489
3	0.87693015
4	0.86667996
5	0.86607560
6	0.86607560

Discussion

In this paper, approximate approaches are proposed to solve QOC problems for both finite and infinite time performance indices depending on Laguerre polynomials and Laguerre functions. The following points can be stated :

- From the numerical results, it is clear that using Laguerre basis function to approximate the states will produce accurate solution as N increases and the numerical solution converges to the correct optimal trajectories as the length of series increases.

- Since the proposed algorithm depending on Laguerre polynomials doesn't deal with the infinite time problems, the finite time version of the infinite time problems can be considered, but the infinite time optimal performance index can be approximated by a finite time optimal performance index if the states $x^*(t_f)$ and $\hat{x}(t_f)$ are near the origin, where $x^*(t_f)$ is the optimal state of the infinite time problem at time $t = t_f$ and $\hat{x}(t_f)$ is the optimal state of the finite time problem at the end time, therefore; the algorithm depending on Laguerre functions avoids the problem associated with the algorithm using Laguerre polynomials to solve all infinite time problems with satisfactory results.

The main advantages of the proposed algorithms are:

+ The OC problem is converted into quadratic programming problem with a few linear constraints, which can be solved using the standard quadratic programming .
+ The number of unknown parameters is kept as small as possible.
+ The coefficients a_{ij} and b_{ij}; $i = 0,1,\dots n$, $j = 0,1,\dots m$ decrease rapidly as N increases.

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الطريقة المباشرة لأمثلية المسار باستخدام دوال لاكير

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الخلاصة

هذا البحث يُقدم طريقة تقريبية لحل مسألة السيطرة المثلى غير المُقيدة. هذه الطريقة تُصنف كطريقة مباشرة و التي تحول مسألة السيطرة المثلى الى مسألة البرمجة الرياضية. الطريقة المباشرة المقترحة أنجزت باستخدام أسلوب المعلمات لمتغيرات الحالة بمساعدة متعددات حدود لاكير ودوال لاكير لتقريب منظومة متغيرات الحالة. أشتقت و برهنت بعض الخواص الجديدة لمتعددات حدود ودوال لاكير لتسهيل الحسابات في هذه الطريقة. بالإضافة إلى ذلك، قد تم أشتقاق شروط الأقتراب للخوارزمية المقترحة لحل مسألة السيطرة المثلى. طبقت الطريقة المطروحة على بعض الأمثلة وأعطيت نتائج أفضل مقارنة مع طرق أخرى.