On $p\theta$-open sets in topological spaces

Nehmat K. Ahmed* **and** ***Sami A.Hussein*****
****College of Scientific Education – University of Salahaddin***
*****College of Basic Education – University of Salahaddin***

Abstract

The aim of this paper is to introduce and study some properties of pre-θ-open sets,and study a new class of spaces, called $p\theta$-regular space. Determine some properties of $p\theta$-regularity and compare with other types of regular spaces.

Introduction

julian Dontchev, Maximilian Ganster and Takashi Noiri (2000) has introduced the concept of $p\theta$-open sets in topological spaces. The purpose of the present paper is to introduced and investige a new separation axiom called pθ-regular space, by using such sets, we have proved that $p\theta$-open and θ-open set are identical in $p\theta$-regular spaces.

Definitions and Preliminaries

By a space X we mean a topological space on which no separation axioms is assumed, we recall the following definitions, notational conventions and characterizations. The colsure(resp., Interior)of a subset A of X is denoted by ClA(resp.,IntA). A subset A of X is said to be preopen(Mashour A.S., Abd-El-Monsef M.E. and El-Deeb S.N.,1982) (resp.,pre-regular p-open (Ganster,1987),θ-open(Velico,1968), pθ-open (Ganster,2000),δ-open(Velico ,1968), pδ-open(Hussein,2003)of a space X, if and only if A⊂InclA(resp., A=pIntApClA,if for each x∈A,there exist an open(resp.,preopen,open,preopen) set G, such that x ∈G⊂ clG⊂ A,(resp., x ∈G⊂ pclG⊂ A, x ∈G⊂ IntClG⊂ A, x ∈G⊂ pIntpClG⊂ A).The family of all preopen(resp.,pre-regular p-open ,θ-open, pθ-open,δ-open, pδ-open) set of a space X,is such that x ∈G⊂ clG⊂ A,(resp., x ∈G⊂ pclG⊂ A , x ∈G⊂ IntClG⊂ A, x ∈G⊂ pIntpClG⊂A).The family of all preopen (resp.,pre-regular p-open ,θ-open, pθ-open,δ-open, pδ-open) set of a space X,is denoted by $PO(X)$(resp.PRPO(X), $\theta O(X)$, $P\theta O(X)$,δO(X), PδO(X)). The complemnet of preopen (resp.,pre-regular p-open,θ-open,pθ-open, δ-open,pδ-open)set is called preclosed(resp.,pre-regular p-closed,θ-closed ,Pθ-closed, δ-closed,pδ-closed) set. The intersection of all preclosed (resp., θ-closed, δ-closed, pδ-closed)sets containing A is called preclosure (resp., θ-closure and $P\theta$–closure, δ–closure, pδ–closure).and is denoted by pclA (resp., $cl_{\theta}A$ and $pcl_{\theta}A$, δ-clA, $pcl_{\delta}A$) The union of all preopen (resp., θ-open, δ-open, pδ-open)sets contained in A is called preinterior (resp., θ-interior and $P\theta$–interior, δ–interior, pδ–interior) .and is denoted by pIntA (resp., θ-IntA and $pInt_{\theta}A$, δ-IntA, $pInt_{\delta}A$) . A space X is said to be submaximal (Ganster,1987**)** if and only if every dense subset of X is open set. A space X is said to be pre-T_2 (Mashour et al,1982) if for each x,y$\in$X, such that x≠y,there exist disjoint preopen sets G,H,such that x $\in$G and y$\in$H. A space X is said to be p^*-regular (Ahmad,1990)(resp., p^{**}-regular) iff for every x$\in$X,and every preclosed set F such that x$\notin$F,there exist disjoint preopen(resp.,open) set G,H such that x $\in$G and F$\subseteq$H.

Definition 1:

Let A be any subset of aspce X. A piont x$\in$X is in the preclosure of A(briefly x $\in$ pclA) (resp., x$\in cl_{\theta}A$, x$\in pcl_{\theta}A$, x$\in$ $pcl_{\delta}A$) if and only if, for each G $\in$PO(X)(resp.,G $\in\tau$, G $\in$PO(X), G $\in$PδO(X)) containing x, A∩G≠φ(resp., A∩ClG≠φ, A∩pClG≠φ, A∩ $pcl_{\delta}G$≠φ). For properties of definition 1 see (Hussein,2003).

Theorem (1):

The following are equivalent about a space X:

1-X is Alexandroff.

2-Any intersection of open sets is open.

3-Any union of closed sets is closed.

Theorem (2): (Ganster,1987**)**

A space X is submaximal iff every preopen set is open.

Theorem (3): (Ahmad,1990**)**

$pCl(G_1xG_2) \subseteq pCl(G_1)xpCl(G_2)$.

Theorem (4): (Dontchev et al, 2000**)**

Let (Y,τ_Y) be a subspace of a space X .A$\subset$Y, if A$\in PO(X)$,then A$\in PO(Y)$.

Theorem (5): (Ganster&Jafari, 2002**)**

If Y$\in PO(X)$and A$\in PO(Y)$,then A$\in PO(X)$.

Some properties of pre-θ-open sets

Lemma 1:

Each pre-θ-open sets can be written as a union of preopen set.

Proof :

Let A$\in P\theta$O (X) then for each x $\in$A there exist B$\in$PO (X) s.t. x$\in$B$\subset$pclB$\subset$ A, then $\cup\{x; x\in A\}\subset \underset{x\in B}{\cup}$ B$\subset$A Therefore A= $\underset{x\in B}{\cup}$ B

Lemma 2:

Any union of $P\theta$-open set is $P\theta$-open**.**

We have the following diagram of implications and any other implication, except these resulting by transitivity can not be add in general

$$\begin{array}{ccccc} \theta O(X) & \longrightarrow & P\theta O(X) & \longrightarrow & PO(X) \\ \downarrow & & \downarrow & & \\ \delta O(X) & \longrightarrow & P\delta O(X) & & \end{array}$$

it is clear that from diagram every pθ-open set is preopen set, and every θ-open set is $P\theta$-open set. but the converse may not be true in general as in the following example shown.

Example 1 : Let X={a, b, c},and τ_1={Φ, X, {a}},τ_2={Φ, X,{a, b}},then PO(X,τ_1)={Φ, {a}, {a, b}, {a, c}, X} and PθO (X,τ_1)={Φ, X}. Also PθO (X,τ_2)=P (X)/{{c}} but θO (X,τ_2)={Φ, X}.

Remark 1:

The intersection of two $P\theta$-open sets need not be $P\theta$-open set in general

Example 2 : Let X={a, b, c}, τ ={Φ, X, {a, b}}, then {a, c}$\in P\theta$O (X) and {b, c}$\in P\theta$O (X) but {a, c}∩{b, c}={c}$\notin P\theta$O (X).

Lemma 3:

θO (X) and PθO(X) are identical if (X,τ) is submaximal.

Proof : From diagram we have every θO(X) is PθO(X),so to show that θO(X) is PθO(X) are identical,we have only to show that PθO(X) is θO(X) . Let A $\in$PθO (X) then for all x $\in$ A, there exist G$\in$PO (X) s.t. x$\in$G$\subset$PclG$\subset$A, but since X is submaximal, we have PO (X)=τ, so that G$\in$τ therefore PclG=clG, it follows that x$\in$G$\subset$clG$\subset$A, hence A$\in$θO (X).

Proposition 1:

If (X,τ) is P^*-regular, then every open set is Pθ-open.

Proof : Let A$\in$τ, we have to show that A$\in$PθO (X), A$\in$PO (X) by theorem 3.2.1 [1]. For each x$\in$A, there exist B$\in$PO (X) such that x$\in$B$\subset$PclB$\subset$A, which implies that A$\in$PθO (X).

Corollary 2: PO (X) and PθO (X) are identical if X is p^*-regular space.

Proposition 2:
Let X_1, X_2 be two topological spaces and $X=X_1 \times X_2$,let $A\in P\theta O(X_i)$ for I=1,2, then $A_1 \times A_2\in P\theta(X_1 \times X_2)$.

Proof : Let $(x_1, x_2)\in A_1 \times A_2$ then $x_1\in A_1$ and $x_2\in A_2$ since A_1, $A2\in P\theta O(X_i)$,there exist preopen sets G_1,G_2 such that $x_1\in G_1\subset pClG_1\subset A_1$ and . $x_2\in G_2\subset pClG_2\subset A_2$,$(x_1,x_2)\in G_1 \times G_2\subset PclG_1 \times PclG_2\subset A_1 \times A_2$ but $G_1 \times G_2\subset pCl(G_1 \times G_2)$, and $pCl(G_1 \times G_2)\subset pClG_1 \times pClG_2$ it follows that $(x_1,x_2) \in G_1 \times G_2\subset Pcl\ (G_1 \times G_2)\subset A_1 \times A_2$ so that $A_1 \times A_2\in P\theta(X_1 \times X_2)$.

Proposition 3:
Let (Y,τ_Y) be a subspace of a space (X,τ). If $A\in Y$ and $A\in P\theta O$ (X) then $A\in PO(Y)$

Proof : Let $A\in P\theta O(X)$, to show that $A\in PO(Y)$ we have $A\in P\theta O(X)$, then for each $x\in A$, there exist $G\in PO(X)$ such that $x\in G\subset Pcl_xG\subset A$,but $G\in PO(X)$ and $G\subset A$, but $A\subset Y$, then $G\subset Y$ so that $G\in PO(Y)$ by theorem(3), hence $G\subset Pcl_xG$,but $G=G\cap Y\subset Pcl_xG\cap Y\subset A\cap Y=A$ so for each $x\in A$,there exist $G\in PO(Y)$such that $x\in G\subset Pcl_yG\subset A$,so that $A\in P\theta O$ (Y).

Proposition 4:
Let (Y,τ_Y) be a subspace of a space X, If $Y\in PO(X)$ and $A\in P\theta O(Y)$,then $A\in P\theta O(X)$.

Proof: follows from theorem (4).

Proposition 5:
A space X is pre-T_2 iff for each $x,y\in X$, such that $x\neq y$,there exist preopen sets G,H,such that $x \in G$ and $y\notin pClG$.

Proof: Obvious.

Theorem (6):
A space X is pre-T_2 if and only if every sengelton set is pre-θ-closed

Proof: (Necessity)

Let H={a},and let $b\notin H$,we have $a\neq b$,since X is pre-T2 by theorem5, there exist a preopen set G such that $b\in G$ and $a\notin pClG$, $pClG\cap H=\varphi$, therefor $b\notin pCl_\theta H$, it follows that H pre-θ-closed set. (Sufficiency) Let $a,b\in X$ such that $a\neq b$,and let H={a},by hypothesis H pre-θ-closed,we have $b\notin pCl_\theta H$,there exist a preopen set G such that $b\in G$,$pClG\cap H=\varphi$,then $a\notin pClG$,$a\in X/\ pClG$, $X/\ pClG\cap G=\varphi$,G and X/ pClG are preopen setswhich containing b,a respectively,therefor X is pre-T_2.

Pθ-regular space

Definition 1:

A space X is said to be Pθ-regular iff for each Pθ-closed set F and a point $x\in X$ such that $x\notin F$,there exist two open sets G and H such that $x\in G$, $F\subset H$and $G\cap H=\Phi$.

Proposition 1:

Each P^{**}- regular space is Pθ-regular.

The converse of the a above lemma is not true in general as the following example show

Example 1: Let X={a, b, c}, and τ={Φ, {a},{b},{a, b},X} then PO (X)= τ, and PθO(X)={ Φ, X}, then X is Pθ-regular which is not P^{**}-regular.

Theorem (1):

For any topological space (X,τ) the following are equivalent:-

i- (X,τ) is Pθ-regular

ii-For every $x\in X$ and every Pθ-open set A containing x there exists an open set B such that $x\in B\subset clB\subset A$.

iii-Every Pθ-closed set F is the intersection of all closed nbd of F.

iv-For every non-empty subset A of X and every Pθ-open subset B of X such that $A\cap B\neq\Phi$,∃ an open sets C of X such that $A\cap C\neq\Phi$ and $clC\subset B$.

v-For every non-empty subset A of X and every Pθ-closed subset F of X such that $A\cap F=\Phi$, there exist two open sets B and C s.t. $A\cap B\neq\Phi$, $B\cap C=\Phi$ and $F\subset C$.

Proof :(i)→(ii) Let A be a Pθ-open set of X containing x, X\A is Pθ-closed subset of X and $x\notin X\backslash A$ by(i) , there exist two open subsets B and C such that $x\in B$, $X\backslash A\subset C$, and $B\cap C=\Phi$, therefore $x\in B\subset X\backslash C\subset A$ hence $x\in B\subset clB\subset cl(X\backslash C)=X\backslash C\subset A$ which implies that $x\in B\subset clB\subset A$

(ii)→(iii): Let F be Pθ-closed , and $x\notin F$, then $x\in X\backslash F$, and X\F is a Pθ-open subset of X, using(ii) there exists an open set B such that $x\in B\subset clB\subset X\backslash F$, hence $F\subset X\backslash clB\subset X\backslash B$ consequently X\B is a closed nbd of F to which x dose not belong , this prove(iii).

(iii)→(iv) let $\Phi\neq A\subset X$,and B be any Pθ-open subset of X s.t. $A\cap B\neq\Phi$, let $x\in A\cap B$, since $x\notin X\backslash B$ is Pθ-closed so there exists a closed nbd of X\B,say E such that $x\notin E$,let $X\backslash B\subset D\subset E$,where D is an open set,then C=X\E and $x\in C$,and $A\cap C\neq\Phi$ also X\D being closed,$clC=cl(X\backslash E)\subset X\backslash D\subset B$,hence $clC\subset B$.

(iv)→(v): let $\Phi\neq A\subset X$,and F be any Pθ-closed subset X such that $A\cap F=\Phi$, then $A\cap X\backslash F\neq\Phi$ and X\F is Pθ-open subset using (iv) there exists an open

subset B of X such that $A\cap B\neq\Phi$ and $B\subset clB\subset X\backslash F$.Putting $C=X\backslash clB$ then $F\subset C\subset X\backslash B$ and C is open, this implies (v).
(v)→(i): Let $x\notin F$ where F is Pθ-closed , and let $A=\{x\}\neq\Phi$,then $A\cap F=\Phi$, and hence using(v) ∃ two open sets B and C such that $A\cap B\neq\Phi$, $B\cap C=\Phi$, and $F\subset C$ which implies that (X,τ) is Pθ-regular.

Proposition 2:
A topological space (X,τ) is Pθ-regular iff for every $x\notin F$, F is Pθ-closed, ∃ two open subsets G and H s.t. $x\in G$ and $F\subset H$ and $clG\cap clH=\Phi$.
Proof: The sufficiency follows directly, and necessity follows from theorem4.1ii.

Proposition 3:
If A is clopen subset of X, then A is Pθ-open set.
Proof: If $A=\Phi$, there is nothing to prove,if $A\neq\Phi$, let $x\in A$, then $x\in A\subset pclA=A\cup clIntA\subset A$.

Theorem 3:
If X is Alexandroff and Pθ-regular space, then every pθ-open set A is clopen set.
Proof: Let A be Pθ-open set, then by (theorem 4.1 ii), there exist an open set G_x such that $x\in G_x\subset clG_x\subset A$, hence $A=\cup\{G_{x,\ x\in A}\}=\bigcup_{x\in A}\{clG_{x,\ x\in A}\}$ it follows that A is open set since X is Alexandroff space union of any closed set is closed $\cup_{x\in A}clG_x$ is closed, then A is closed. Therefore A is closed as well as open.

Theorem 4:
If X is Pθ-regular space then $P\theta O(X)=\theta O(X)$.
Proof: Let $A\in P\theta O(X)$,since X is Pθ-regular,by theorem(4.1ii)for each $x\in A$, There exist an open set G such that $x\in G\subset clG\subset A$, therefor $A\in\theta O(X)$
The converse part follows from above diagram

Theorem 5:
A space X is Pθ-regular if $PO(X,\tau)=\tau$.
Proof: It is not hard and therefore it is omitted.

Theorem 6:
Every Pθ-regular and pT_o-space (X,τ) is a T_2-space.
Proof: Let x, $y\in X$,such that $x\neq y$, since X is preT_o-space, then, there exist a preopen set A containing x but not y, A is Pθ-open set containing x but not y, since X is Pθ-regular and $x\in A$, ∃ an open set B s.t. $x\in B\subset clB\subset A$.

Hence B and X\clB are open subsets of X such that $x \in B$, $y \in X \backslash clB$ and $B \cap X \backslash clB = \Phi$ which implies that X is a T_2-space.

Theorem 7:
If XxY is Pθ-regular then both X and Y are Pθ-regular.

Proof: Let $x \in A$, where A is Pθ-open subset of X, then for every $y \in Y$, $(x,y) \in AxY$ where AxY is Pθ-open subset of XxY(by Theorem4.1)usingPθ-regularity of XxY $\exists$an open set G of XxY Such that$(x,y) \in G \subset cl_{XxY}G \subset GxY$ Putting G=UxV where U and V are open set in XxY respectively.Then $(x,y) \in UxV \subset cl_{XxY}Uxcl_{XxY}V \subset AxY$, therefore $x \in U \subset cl_{XxY}U \subset A$, then X is Pθ-regular. Similarly we can prove that Y is Pθ-regular.

References

- Ahmad N.K., (1990): On some types of separation axioms. M.Sc. Thesis college of Science, University of Salahaddin.
- Alexandroff P.S.,Diskrete Rume Rec,.(1937): Math.Sbornik, pp.501-518.
- Dontchev J., Ganster M and Noiri.T,. (2000): On p-closed spaces.Internat. J.Math. Math.Sci.Vol.24, pp.203-212.
- Ganster M., (1987): preopen and resolvable spaces. Kyunpook Math.J., Vol.27, pp.135-143.
- Ganster M, and Jafari S., (2002): On PC-compact spaces. Mem. Fac.Sci. Kochi.univ.ser A (Math) Vol.23, pp. 19-26.
- Hussein S.A., (2003): Application of pδ-open sets in topological spaces. M.Sc. Thesis, College of Education, Univ. of Salahaddin.
- Mashour A.S., Abd-El-Monsef M.E. and El-Deeb S.N.,(1982): On precontinuous and weak precontinuous mapping, proc. Math. Phys. Soc. Egypt, Vol.53, pp.42-52.
- Navalagi G.B., Further properties of pre-T_0, pre-T_1 and pre-T_2 spaces, topology Atlas preprint, #428.
- Velico N.V., (1968): H-closed topological spaces, Amer. Math. Soc. Trans Vol.2, pp.103-118.

Journal of Kirkuk University –Scientific Studies , vol.2, No.3,2007

الفضاء التبولوجي في المجموعة المفتوحة ـــ $p\theta$

نعمت خضر أحمد*   سامي علي حسين**

***كلية التربية ـــ جامعة صلاح الدين**

****كلية التربية الاساس ـــ جامعة صلاح الدين**

الخلاصة

هدف من هذا البحث هو تقديم ودراسة المجموعات المفتوحة من النمط P θ، وادخال صنف جديد من الفضاءات المنتضمة سميتregular -P θ ، دراسة بعض خواصها ومقارنتها مع الفضاءات الاخرى .