Solving a system of fredholm integral equations of the second kind by using power functions.

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Abstract

In this paper we used power functions for finding approximate solution of a system of fredholm integral equations of the second kind (S.F.I.E.2st. K.) with degenerate kernels. And also suggested an algorithm for this method the computer programming is given for the algorithm. The method and algorithm are tested on several numerical examples. After comparing the results with exact solution , it occurred that the results are good.

Introduction

The integral equations are not only importance to mathematicians but also to other sciences like engineering, physics…since some problems their mathematical representation appears directly in terms of differential equations. Other problem whose direct representation is in term of integral equation and their auxiliary conditions, may also be reduced to integral equation(Jerri, 1985). The theory and application of integral equations is an important subject in applied mathematics. Integral equations are used as mathematical models for many and varied physical situations, and integral equations also occur as reformulation of other mathematical problems, (Atkinson, 1997).

In this paper we used drive a spectral method which will be used to solve a system of fredholm integral equations of the second kind with degenerate kernels by using power functions. Sulaiman, N.A. in (1999) solved a 2×2 system of liner (F.I.E.2st. K.) using trapezoid method (Sulaiman, 1999). Saeed, R.K. in (2006) solved a 2×2 system of liner (V.I.E.2st. K.) using power function (Saeed, 2006). Ameen in (2007) solved a system of Non-liner (V.I.E.2st. K.) (Ameen, 2007).

We use explanation method to approximate the solution of (S.F.I.E.2st. K.) with degenerate kernels, since one of the uses is to replace complicated function by some simpler function so that integral operator can be more easily performed. The approximate methods provide analytical procedure for obtaining solution in the form of functions. Which are closed in some sense to the exact solution of the given problem.

The basic idea is to assume that the unknown function $u_i(x), i = 1,2,3,...,m$ can be approximate by some of $n+1$ basic function $\beta_i(x)$:

$$u_i(x) = \sum_{l=0}^{N} a_{il}\beta_l(x), i = 0,1,2,3,...,m$$

Where $\beta_i(x)$ naturally be choosing lineally independent. In this work choose $\beta_l(x) = x^l, l = 0,1,2,3,...,n$. Let the function $u_i(x)$ is a linear combination of $\beta_i(x)$, $i = 0,1,2,3,...,m$ its expansion coefficients $a_{i0}, a_{i1}, a_{i2},..., a_{in}$ $, i = 1,2,3,...,m$ are to be determined uniquely (Ameen, 2007; Sulaiman, 1999**).**

Definition (1). Orthogonal function. (Froberg, 1985).

A set of function $\{\beta_i(x)\}, i = 0,1,...$ is said to be orthogonal for the interval $[a,b]$ with respect to weight function $w(x) \geq 0$ if

$$\int_a^b w(x)\beta_i(x)\beta_j(x)dx = 0 \qquad , \text{for } i \neq j$$

$$\int_a^b w(x)\beta_i(x)\beta_j(x)dx = \alpha_k > 0 \quad , \text{for } i = j$$

If, in addition $\alpha_k = 1$ for each $k = 0,1,...,n$ then the set is said to be orthonormal.

A special case of orthogonal function consists of the set of orthogonal polynomial $\{\beta_n(x)\}$,where n denotes the degree of the polynomials $\beta_n(x)$.

Definition (2). (Power function).(Faires, 2003).

A function $\beta_l(x) = x^l, l = 0,1,2,3,...,n$ is called a power functions.

Definition (3). (Degenerated kernel). (Atkinson, 1997)

Suppose that $k(x,t)$ is a kernel defined on the square $[a,b]*[a,b]$ and there are a finitely many functions $a_1, a_2,..., a_n, b_1, b_2,..., b_n$ such that

$$k(x,t) = \sum_{i=1}^{n} a_i(x)b_i(t) \qquad (a \leq x,t \leq b)$$

In this case the kernel is said to be degenerated kernel.

Definition (4). (S.F.I .E.2st. K.). (Kythe, 2002)

The general form of (S.F.I .E.2st. K.) is as follows.

$$u_i(x) = f_i(x) + \sum_{j=1}^{m}\int_a^b k_{ij}(x,t)u_j(t)dt \qquad ; x \in [a,b], i = 1,2,3,...,m$$

Where $f_i(x), i=1,2,3,...,m, m\in N$ are known continuous functions and $k_{ij}(x,t); i, j=1,2,3,...,m$ are given continuous functions on $[a,b]$ while $u_i(x), i=1,2,3,...,m$ are unknown functions to be determined are called (S.F.I.E.2st. K.).

Technique which is used power function to solve a (S.F.I.E.2st. K.) with degenerate kernels

$$u_i(x)= f_i(x)+\sum_{j=1}^{m}\int_a^b k_{ij}(x,t)u_j(t)dt \quad ; x\in[a,b], i, j=1,2,3,...,m \qquad \dots (1)$$

assume that approximate $u_i(x), i=1,2,3,...,m$ by a finite length series $\{\beta_i(x)\}_{i=0}^{\infty}$; that's mean

$$u_i(x)=\sum_{l=0}^{n} a_{il}\beta_l(x), i=1,2,3,...,m \qquad \dots (2)$$

Substation (2) in (1) we get

$$\sum_{l=0}^{n} a_{il}\beta_l(x)= f_i(x)+\sum_{j=1}^{m}\int_a^b k_{ij}(x,t)\sum_{l=0}^{n} a_{il}\beta_l(t)dt \quad ; , i=1,2,3,...,m \qquad \dots (3)$$

then we get

$$\sum_{l=0}^{n} a_{il}\beta_l(x)= f_i(x)+\sum_{j=1}^{m}\sum_{l=0}^{n} a_{il}\int_a^b k_{ij}(x,t)\beta_l(t)dt \; ; , i=1,2,3,...,m \qquad \dots (4)$$

since by definition (2) we have $\beta_l(x)=x^l, l=0,1,2,3,...,n$ then we get:

$$\sum_{l=0}^{n} a_{il}x^l = f_i(x)+\sum_{j=1}^{m}\sum_{l=0}^{n} a_{il}\int_a^b k_{ij}(x,t)t^l dt \; ; , i=1,2,3,...,m \qquad \dots (5)$$

$$\sum_{l=0}^{n} a_{il}x^l -\sum_{j=1}^{m}\sum_{l=0}^{n} a_{il}\int_a^b k_{ij}(x,t)t^l dt = f_i(x) \; ; , i=1,2,3,...,m \qquad \dots (6)$$

From equation (6), we get $L\times L$ system of algebraic equations where $L=m\times(n+1)$ as follows

For each $, i=1,2,3,...,m$, equate the coefficients up to x^n, we get the square system of algebraic equations for $a_{i0}, a_{i1}, a_{i2}, ..., an \quad , i=1,2,3,...,m$.

We solve the resulting system by using Gauss elimination method to find $a_{i0}, a_{i1}, a_{i2}, ..., an , i=1,2,3,...,m$. Then we put the values of $a_{i0}, a_{i1}, a_{i2}, ..., an$ $, i=1,2,3,...,m$ in equation (2), and we get approximate solution of equation (1).

Algorithm of the technique

Step 1: let i=1 in equation (2) to determine $u_1(x)$.

Step 2: Equate the coefficients up of x^i , , $i=1,2,3,...,n$ from the lift –hand- side to the coefficients from the right –hand- side.

Step 3: From a system for , $i=2,3,...,m$ determine $u_i(x)$.

Step4: We get the system of L unknown equations $a_{il}(x), i=1,2,3,...,m; l=1,2,3,...,n$.

Step 5: Compute the value of all integrals from the resulting system by using methods of integrations.

Step 6: Put the values of $a_{il}(x), i=1,2,3,...,m; l=1,2,3,...,n$ .in equation (2) we Obtain the approximate solution of $u_i(x), i=1,2,3,...,m$.

Numerical examples and results

The following examples are solved by above technique.

Example (1):- Consider a system of fredholm integral equations of the second kind with degenerate kernels

$$u_1(x)=e^x-\frac{2x}{3}-1+\int_0^1(3+\frac{8xt}{3})u_2(t)dt$$

$$u_2(x)=x^2-x+\int_0^1(xt)u_1(t)dt$$

With the exact solution $u_1(x)=e^x$ and $u_2(x)=x^2$.

Sol.:- Apply the Numerical technique in section (3) for n=2 and m=2 we obtained

$u_i(x)=\sum_{l=0}^{n} a_{il}\beta_l(x), i=1,2.$ Where $\beta_l(x)=x^l, l=1,2,3,...,n$ which is approximate solution of a system. Substituted these in a system. After that compute the single integrations by using the methods of integration or numerical integration, equate the coefficients we get (2×3) a liner system of equations. Then use Gauss elimination method to find the value of coefficients .we get

$$a_{10}=1, a_{11}=1, a_{12}=\frac{1}{2}, a_{20}=0, a_{21}=0, a_{22}=1$$

The results for example (1) is shown in table (1).

Table (1): Show a comparison between the exact solution and the approximate results (numerical solution).Obtained by above technique.

	$u_1(x)=e^x$		$u_2(x)=x^2$	
x	**Exact values**	**Approximate values**	**Exact values**	**Approximate values**
0.00	1.0000000	1.000000	0.0000000	0.0000000
0.10	1.1051700	1.105000	0.0100000	0.0100000
0.20	1.2211402	1.220000	0.0400000	0.0400000
0.30	1.3449858	1.345000	0.0900000	0.0900000
0.40	1.4918246	1.480000	0.1600000	0.1600000
0.50	1.6487212	1.625000	0.2500000	0.2500000
0.60	1.8221188	1.780000	0.3600000	0.3600000
0.70	2.0013752	1.945000	0.4900000	0.4900000
0.80	2.2255409	2.120000	0.6400000	0.6400000
0.90	2.4596031	2.305000	0.8100000	0.8100000
1.00	2.7182818	2.500000	1.0000000	1.0000000
L.S.E.	$1.24200*10^{-6}$		0.00000000	

Example (2):- Consider a system of fredholm integral equations of the second kind with degenerate kernels

$$u_1(x)=2x-2e^x+\int_0^1 e^{x-t}u_2(t)dt$$

$$u_2(x)=2e^x-\frac{2x}{3}-1+\int_0^1 (1+xt)u_1(t)dt$$

with the exact solution $u_1(x)=2x$ and $u_2(x)=2e^x$.

Sol.:- Apply the Numerical technique in section (3) for n=2 and m=2 we obtained

$u_i(x)=\sum_{l=0}^{N} a_{il}\beta_l(x), i=1,2.$ Where $\beta_l(x)=x^l, l=1,2,3,...,n$ which is approximate solution of a system. Substituted these in a system. After that compute the single integrations by using the methods of integration or numerical integration , equate the coefficients we get (2×3) a liner system of equations. Then use Gauss elimination method to find the value of coefficients .we get

$a_{10}=0, a_{11}=2, a_{12}=0, a_{20}=2, a_{21}=2, a_{22}=1$

The result for example (2) is shown in table (2).

Table (2): shows a comparison between the exact solution and the approximate results (numerical solution).Obtained by above technique.

	$u_1(x) = 2x$		$u_2(x) = 2e^x$	
x	**Exact values**	**Approximate values**	**Exact values**	**Approximate values**
0.00	0.000000	0.000000	2.000000000	2.000000
0.1	0.200000	0.200000	2.210241836	2.210000
0.2	0.400000	0.400000	2.442805506	2.440000
0.3	0.600000	0.600000	2.699717616	2.690000
0.4	0.800000	0.800000	2.882649396	2.860000
0.5	1.000000	1.000000	3.297395942	3.250000
0.6	1.200000	1.200000	3.640960000	3.560000
0.7	1.400000	1.400000	4.026505414	3.890000
0.8	1.600000	1.600000	4.450261334	4.240000
0.9	1.800000	1.800000	4.817516500	4.610000
1.00	2.000000	2.000000	5.523333340	5.000000
L.S.E.	0.000000		$2.33000 * 10^{-5}$	

Conclusion

The approximate or numerical solution for a system of (S.F.I.E.2st. K.) with degenerate kernels is introduced by using a power functions. Algorithm of the technique tested on several Numerical examples, after testing the technique, the results which are obtained in tables (1, 2), indicate good. Also, comparison is made between numerical solution and exact solution dependent on least square error (L.S.E.), which is calculated from the numerical solution against the exact solution.

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حل نظام معادلات فريدهولم التكاملية من النوع الثاني باستخدام دوال القوى

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الخلاصة

في هذا البحث تم استخدام دوال القوى لايجاد حل تقريبي لنظام معادلات فريدهولم التكاملية من النوع الثاني ذات النوات المحللة، ايضا تم اقتراح خوارزمية لهذه الطريقة و برمجتها واختبرت هذه الخوارزمية على الامثلة العددية، و قارنا النتائج مع الحل المضبوط وكانت جيدة.