

Approximate Solution for the System of Nonlinear Volterra Integral Equations of the Second Kind by Weighted Residual Methods

Rostam K. Saeed Chinar S. Ahmed***

** College of Science - University of Salahaddin/Erbil*

*** College of Science - University of Sulaimani*

Received:2007/4/11, Accepted:2008/4/8

Abstract

In this paper, three types of weighted residual methods (Collocation, Subdomain, and Galerkin methods) are presented for finding an approximate (sometimes exact) solution of the system of non-linear Volterra integral equations of the second kind (VIEK2). We showed the efficiency of the prescribed methods by solving some numerical examples.

Introduction

We use expansion method to approximate the solution of the system of non-linear VIEK2 since one of its uses is to replace complicated functions by some simpler functions so that integral operations can be more easily performed. The approximate methods provide analytical procedure for obtaining solutions in the form of functions, which are close in some sense to the exact solution of the given problem (Boyd, 2000).

The basic idea is to assume that the unknown functions $u_i(x), i = 1, 2, \dots, m$ can be approximated by sum of $N+1$ "basis functions" $\Psi_k(x)$:

$$S_{iN}(x) = \sum_{k=0}^N a_{ik} \Psi_k(x) \quad , \quad i = 1, 2, \dots, m \quad \dots(1)$$

where $\Psi_k(x)$ naturally be choosing linearly independent (in this work, we choose $\psi_k(x) = x^k, k=0, 1, \dots, N$). By assuming the function $S_{iN}(x)$ is a linear combination of $\Psi_k(x)$, it's expansion coefficients $a_{i0}, a_{i1}, \dots, a_{iN}, i = 1, 2, \dots, m$ are to be determines uniquely, see (Delves, 1974), (Polyanin, 1998). To strengthen the expansion method some error minimizing techniques are needed for determining the coefficients is $S_{iN}(x)$. One of the popular techniques is the weight residual methods which include the (Collocation, Subdomain and Galerkin method). A weighted residual methods (WRM's) has been used for solving differential equations and integral equations by many authors and researchers (Chapra & Canale, 2002; Davis, 1975; Hall & Watt, 1976; Chambers, 1976; Atkinson,

1992; Babolian & Biazar, 2000; Brunner, 1990; Brunner et. al, 1999; Kuthen & Brunner, 1997 and Mahmoudi, 2005). On the other hands (Saeed, 2006) used this method to solve system of linear VIEK2 and system of linear Volterra integro-differential equations of the second kind.

In this paper we aim to obtain an approximate solutions for a system of nonlinear VIEK2 by using WRM's.

The rest of this paper is organized as follows:

In section 2, WRM's is reformulated to be suitable for a system of Non-linear VIEK2's.

In section 3, WRM's has been presented to solve a system of Non-linear VIEK2.

In section 4, numerical results are given for illustrations, depending on the least square errors.

Formulation of Weighted Residual Methods for Solving System of Non-linear VIEK2

In this section, we endeavor to reformulate the WRM's to solve non-linear system of VIEK2 as follows:

Consider the functional equation given by

$$L_i[u_i(x)] = f_i(x), \quad x \in D, \quad i = 1, 2, \dots, m, \quad \dots(2)$$

where $L_i : C[a, b] \rightarrow C[a, b]$ operators of the integral equations defined by

$$L_i[u_i(x)] = u_i(x) - \int_a^x k_i(x, t, u_i(t)) dt, \quad i=1, 2, \dots, m \quad \dots(3)$$

Substituting the approximate solution $S_{iN}(x)$ given by (1) into equation (2), the result is the so-called "residual function" defined by

$$E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN}) = L_i[S_{iN}(x)] - f_i(x), \quad i = 1, 2, \dots, m \quad \dots(4)$$

Since the residual function $E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN})$ is identically equal to zero for the exact solution, the challenge is to choose the coefficients $a_{i0}, a_{i1}, \dots, a_{iN}$ so that the residual function is minimized. Throughout this study we shall endeavor to minimize $E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN})$ in some sense (Saeed, 2006). The goal of WRM's is to choose the coefficients $a_{i0}, a_{i1}, \dots, a_{iN}$ so that the residue $E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN})$ becomes small (in fact zero) over a chosen domain. In integral form this can be achieved with the condition

$$\int_D w_i(x) E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN}) dx = 0, \quad \dots(5)$$

where $w_i(x)$ are prescribed weight function, the technique described by (5) is called WRM's, by which the optimal values of $a_{i0}, a_{i1}, \dots, a_{iN}$ that minimize $E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN})$, is determined.

We now presented three methods of the WRM's to determine $a_{i_0}, a_{i_1}, \dots, a_{i_N}, i = 1, 2, \dots, m$ in the equation (1):

1. Collocation Method:

Is one of the methods which can be used to evaluate the parameters ; $a_{i_0}, a_{i_1}, \dots, a_{i_N}; i = 1, 2, \dots, m$ that minimize the residual function $E_{iN}(x, a_{i_0}, a_{i_1}, \dots, a_{i_N}), i = 1, 2, \dots, m$. The main idea behind this method is the parameters $a_{i_0}, a_{i_1}, \dots, a_{i_N}; i = 1, 2, \dots, m$ are to be found by forcing that the residual $E_{iN}(x, a_{i_0}, a_{i_1}, \dots, a_{i_N})$ vanishes at given set of $N+1$ points in the domain D . Mathematically, this can be described as follows:

Let us choose $N+1$ distinct nodes $x_0, x_1, \dots, x_N \in D$ and defined the weighted functions be Dirac delta functions, $w_j(x) = \delta(x - x_j), j = 0, 1, \dots, N$ which vanishes everywhere except at $x = x_j, j = 0, 1, \dots, N$. This means that

$$\delta(x - x_j) = \begin{cases} 0 & \text{if } x \neq x_j \\ 1 & \text{if } x = x_j \end{cases} \quad \text{for } j = 0, 1, \dots, N$$

Then the equation (4), become

$$\int_D \delta(x - x_j) E_{iN}(x, a_{i_0}, a_{i_1}, \dots, a_{i_N}) dx = 0,$$

this can be written as

$$E_{iN}(x, a_{i_0}, a_{i_1}, \dots, a_{i_N}) = 0, i = 1, 2, \dots, m; j = 0, 1, \dots, N \quad \dots(6)$$

Moreover the distribution of the collocation nodes on D is arbitrary; however, in practice we describe it uniformly on D .

The equation (6) will provide us by $m \times (N + 1)$ simultaneous equations to determine the parameters $a_{i_0}, a_{i_1}, \dots, a_{i_N}; i = 1, 2, \dots, m$.

2. Subdomain Method:

It is another method which can be used to calculate the parameters; $a_{i_0}, a_{i_1}, \dots, a_{i_N}; i = 1, 2, \dots, m$. Here the domain D is divided in $N+ 1$ sub-domain $D_j, j = 0, 1, \dots, N$ where

$$w_j(x) = \begin{cases} 1 & \text{if } x \in D_j \\ 0 & \text{if } x \notin D_j \end{cases}$$

This method, minimize the residual error in each of the chosen sub-domains. Hence the equation (2) is satisfied in each of the sub-domains D_j , therefore equations (4) become

$$\int_{D_j} E_{iN}(x, a_{i_0}, a_{i_1}, \dots, a_{i_N}) dx = 0, j = 0, 1, \dots, N. \quad \dots(7)$$

Note that the choice of the sub-domains is free. In many cases an equal division of the total domain is likely the best choice. However, if higher

resolution (and a corresponding small error) in a particular area is desired, a non-uniform choice may be more appropriate.

3. Galerkin Method:

Galerkin approach makes the residual $E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN})$ orthogonal to $N+1$ given linearly independent function on the domain D . In this approach the weighting function $w_j(x)$ is chosen to be identical to the base functions, i.e.

$$w_j(x) = \frac{\partial S_{iN}(x)}{\partial a_{ik}} ; k = 0, 1, 2, \dots, N , i = 1, 2, \dots, m$$

where $S_{iN}(x)$ is the approximated solution of the problem, therefore by this approach the weighted integral of $E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN})$ become

$$\int_D \frac{\partial S_{iN}(x)}{\partial a_{ik}} E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN}) dx = 0, k = 0, 1, 2, \dots, N . \dots(8)$$

This will provide $m \times (N+1)$'s non-linear simultaneous equations to determine the parameters $a_{i0}, a_{i1}, \dots, a_{iN}, i = 1, 2, \dots, m$.

Solution of a System of Non-linear VIEK2 by Weighted Residual Method

In this section we approximate the solution of a system of non-linear VIEK2

$$U(x) = F(x) + \int_a^x K(x, t, U(t)) dt , \dots(9)$$

where

$$U(x) = (u_1(x), \dots, u_m(x))^T, U(t) = (u_1(t), u_2(t), \dots, u_m(t)),$$

$$F(x) = (f_1(x), \dots, f_m(x))^T,$$

$$K(x, t, U(t)) = (k_1(x, t, U(t)), \dots, k_m(x, t, U(t)))^T,$$

by means of the WRM's, described in section 2. The aim is to find the parameter coefficients $a_{i0}, a_{i1}, \dots, a_{iN}, i = 1, 2, \dots, m$.

Using operator's form (2), system (9) can be written as follows:

$$L_i[u_i(x)] = u_i(x) - \int_a^x K_i(x, t, U(t)) dt , i = 1, 2, \dots, m \dots(10)$$

Where the unknown functions $u_i(x)$ are approximated by $S_{iN}(x)$ which is given by equation (1).

Now the approximate solution (1) substituting in the system (10) to obtain:

$$L_i[S_{iN}(x)] = \sum_{k=0}^N a_{ik} x^k - \int_a^x K(x, t, \sum_{k=0}^N a_{1k} t^k, \sum_{k=0}^N a_{2k} t^k, \dots, \sum_{k=0}^N a_{mk} t^k) dt$$

From equation (4) we obtain the following residual equations

$$E_{iN}(x, a_{ik}; k = \overline{0, N}) = \sum_{k=0}^N a_{ik} x^k - \int_a^x K_i(x, t, \sum_{k=0}^N a_{1k} t^k, \sum_{k=0}^N a_{2k} t^k, \dots, \sum_{k=0}^N a_{mk} t^k) dt - f_i(x),$$

$$i = 1, 2, \dots, m. \quad \dots(11)$$

From equation (11) with letting $a_{i0} = f_i(a)$, $i = 1, 2, \dots, m$ we get

$$E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN}) = \sum_{k=0}^N a_{ik} x^k - f_i(x) - \int_a^x K(x, t, \sum_{k=0}^N a_{1k} t^k, \sum_{k=0}^N a_{2k} t^k, \dots, \sum_{k=0}^N a_{mk} t^k) dt,$$

$$i = 1, 2, \dots, m \quad \dots(12)$$

Now, the problem is how to find the optimal values of a_{i1} , a_{i2} , ..., a_{iN} which minimize the residual $E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN})$ in the equation (12), this can be achieved by using the WRM's as follows:

1. Collocation Method:

This method presents the $V=m \times N$ conditions by applying the equation (6) on the residue in equation (12), gives

$$E_{iN}(x_{\beta}, a_{i0}, a_{i1}, \dots, a_{iN}) = \sum_{k=0}^N a_{ik} (x_{\beta})^k - f_i(x_{\beta}) - \int_a^{x_{\beta}} K_i(x_{\beta}, t, \sum_{k=0}^N a_{1k} t^k, \sum_{k=0}^N a_{2k} t^k, \dots, \sum_{k=0}^N a_{mk} t^k) dt = 0 \quad \dots(13)$$

where $x_{\beta} = \beta h, \beta = 0, 1, 2, \dots, V$ and h is to be chosen.

Equation (13) leads to the following non-linear system of equations

$$\sum_{k=0}^N a_{ik} (x_{\beta})^k - \int_a^{x_{\beta}} K_i(x_{\beta}, t, \sum_{k=0}^N a_{1k} t^k, \sum_{k=0}^N a_{2k} t^k, \dots, \sum_{k=0}^N a_{mk} t^k) dt = f_i(x_{\beta}), \beta = 1, 2, \dots, V \quad \dots(14)$$

Here, we need to determine V coefficients $a_{lq}, l = 1, 2, \dots, m, q = 1, 2, \dots, N$. We construct $V \times V$ system of non-linear simultaneous equations from equation (14) to find the coefficient $a_{i1}, a_{i2}, \dots, a_{iN}$. Solve the resulting non-linear system by using modified Newton-Raphson method for the coefficients $a_{i1}, a_{i2}, \dots, a_{iN}$ and substitute the values of the coefficients into the equation (1), we obtain the approximate solution of (9).

The algorithm (VNPFCM):

Step (1): Let $h = \frac{b-a}{n}$, where $n = V + 1$ to find $x_{\beta} = x_0 + \beta h$ for $\beta = 0, 1, \dots, V$.

Step (2): Find $\int_a^{x_{\beta}} K_i(x_{\beta}, t, f_1(a) + \sum_{k=1}^N a_{1k} t^k, f_2(a) + \sum_{k=1}^N a_{2k} t^k, \dots, f_m(a) + \sum_{k=1}^N a_{mk} t^k) dt,$
 $i = 1, 2, \dots, m; \beta = 1, 2, \dots, V$.

Step (3): Substitute the results in step (2) into the equation (14).

Step (4): Solve resulting system, by using modified Newton-Raphson method to find the unknowns $a_{ik}, i = 1, 2, \dots, m, k = 1, \dots, N$ then

substitute the values of a_{ik} , into the equation (1) to get the approximate solution of (9).

2. Subdomain Method:

The V non-linear equations in $a_{i1}, a_{i2}, \dots, a_{iN}, i=1,2,\dots,m$ are obtained by assuming the integration of the residual $E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN})$ given by the equation (7) over the intervals $[x_\beta, x_{\beta+1}], \beta = 0,1, \dots, V-1$ are vanishing, i.e.

$$\int_{x_\beta}^{x_{\beta+1}} E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN}) dx = 0. \quad \dots(15)$$

Substituting $E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN})$ in the equation (12) into the equation (15) yields:

$$\int_{x_\beta}^{x_{\beta+1}} \left[f(a) + \sum_{k=1}^N a_{ik} x^k - \int_a^x K(x, t, f_1(a) + \sum_{k=1}^N a_{1k} t^k, f_2(a) + \sum_{k=1}^N a_{2k} t^k, \dots, f_m(a) + \sum_{k=1}^N a_{mk} t^k) dt \right] - \int_a^{x_\beta} f_i(x) dx = 0. \quad \dots(16)$$

From equation (16) we get the following system of non-linear simultaneous equations

$$\sum_{k=1}^N \int_{x_\beta}^{x_{\beta+1}} a_{ik} x^k - \int_{x_\beta}^{x_{\beta+1}} \int_a^x K_i(x, t, f_1(a) + \sum_{k=1}^N a_{1k} t^k, f_2(a) + \sum_{k=1}^N a_{2k} t^k, \dots, f_m(a) + \sum_{k=1}^N a_{mk} t^k) dt - \int_{x_\beta}^{x_{\beta+1}} (f_i(x) - f_i(a)) dx, \beta = 0,1, \dots, V-1 \quad \dots(17)$$

Solve the above non-linear system for $a_{i1}, a_{i2}, \dots, a_{iN}, i=1,2,\dots,m$ by using modified Newton-Raphson method and substitute the coefficients into the equation (1), we obtain the approximate solution of (9).

The Algorithm (VNPFSM)

Step (1): Let $h = \frac{b-a}{n}$, where $n = V+1$ to find $x_\beta = x_0 + \beta h$ for $\beta = 0,1, \dots, V$.

Step (2): Find $\sum_{k=0}^N \int_{x_\beta}^{x_{\beta+1}} a_{ik} x^k dx$ for $i=1,2,\dots,m; \beta = 0,1, \dots, V-1$.

Step (3): Find $\int_{x_\beta}^{x_{\beta+1}} \int_a^x K_i(x, t, f_1(a) + \sum_{k=1}^N a_{1k} t^k, f_2(a) + \sum_{k=1}^N a_{2k} t^k, \dots, f_m(a) + \sum_{k=1}^N a_{mk} t^k) dt dx$ for $i=1,2,\dots,m; \beta = 0,1, \dots, V-1$.

Step (4): Find $\int_{x_\beta}^{x_{\beta+1}} (f_i(x) - f_i(a)) dx$ for $i=1,2,\dots,m; \beta = 0,1, \dots, V-1$.

Step (5): Substitute the results in steps (2)-(4) into the equation (17).

Step (6): Solve the resulting system, by using modified Newton-Raphson method to find unknowns a_{ik} , and then substitute the values of a_{ik} into (1) to get the approximate solution of (9).

3. Galerkin Method:

The V non-linear equations in $a_{i1}, a_{i2}, \dots, a_{iN}, i=1,2,\dots,m$ are obtained by assuming the integration of the residual $E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN})$ which defined by the equation (8) over the intervals $[a, x_\beta], \beta=1,2,\dots,V$ are vanishing, i.e.

$$\int_a^{x_\beta} \frac{\partial S_{iN}(x)}{\partial a_{ik}} E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN}) dx = 0. \quad \dots(18)$$

Substituting $E_{iN}(x, a_{i0}, a_{i1}, \dots, a_{iN})$ in the equation (12) into the equation (18) yields:

$$\int_a^{x_\beta} x^\beta \left[f_i(a) + \sum_{k=1}^N a_{ik} x^k - \int_a^x K_i(x,t, f_1(a) + \sum_{k=1}^N a_{1k} t^k, f_2(a) + \sum_{k=1}^N a_{2k} t^k, \dots, f_m(a) + \sum_{k=1}^N a_{mk} t^k) dt \right] dx - \int_a^{x_\beta} x^\beta f_i(x) dx = 0. \quad \dots(19)$$

From (19) we get the following non-linear system of equations

$$\int_a^{x_\beta} x^\beta \left(\sum_{k=1}^N a_{ik} x^k \right) dx - \left\{ \int_a^{x_\beta} \int_a^x K_i(x,t, f_1(a) + \sum_{k=1}^N a_{1k} t^k, f_2(a) + \sum_{k=1}^N a_{2k} t^k, \dots, f_m(a) + \sum_{k=1}^N a_{mk} t^k) dt \right\} dx = \int_a^{x_\beta} x^\beta (f_i(x) - f_i(a)) dx, \quad \beta = 1, 2, \dots, V \quad \dots(20)$$

Solve the above non-linear system for $a_{i1}, a_{i2}, \dots, a_{iN}, i=1,2,\dots,m$ by using modified Newton-Raphson method and substitute the coefficients into the equation (1), we obtain the approximate solution of (9).

The Algorithm (VNPFGM)

Step (1): Let $h = \frac{b-a}{n}$, where $n = V + 1$ to find $x_\beta = x_0 + \beta h$ for $\beta = 1, 2, \dots, V$.

Step (2): Find $\int_a^{x_\beta} x^\beta \left(\sum_{k=1}^N a_{ik} x^k \right) dx$ for $i = 1, 2, \dots, m; \beta = 1, 2, \dots, V$.

Step (3): Find

$$\int_a^{x_\beta} x^\beta \left[\int_a^x K_i(x,t, f_1(a) + \sum_{k=1}^N a_{1k} t^k, f_2(a) + \sum_{k=1}^N a_{2k} t^k, \dots, f_m(a) + \sum_{k=1}^N a_{mk} t^k) dt \right] dx$$

for $i = 1, 2, \dots, m$; $\beta = 1, 2, \dots, V$.

Step (4): Find $\int_a^{x_\beta} x^\beta (f_i(x) - f_i(a)) dx$ for $i = 1, 2, \dots, m$; $\beta = 1, 2, \dots, V$.

Step (5): Substitute the results in step (2)-(4) into the equation (20).

Step (6): Solve the resulting system, by using modified Newton-Raphson method to find the unknowns a_{ik} , $k = 1, 2, \dots, N$ and then substitute the values of a_{ik} in the equation (1) to get the approximate solution of (9).

Illustrative Examples

In order to assess both the applicability and accuracy of the theoretical results in section 3, we have applied them to a variety of non-linear system of VIEK2's in the following examples:

Example 1: (Babolian, 2000)

The non-linear system of VIEK2

$$u_1(x) = x - x^2 + \int_0^x (u_1(t) + u_2(t)) dt$$

$$u_2(x) = x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \int_0^x (u_1^2(t) + u_2(t)) dt$$

have the following exact solutions

$$u_1(x) = x \quad \text{and} \quad u_2(x) = x .$$

Solution: Assume that the approximate solution:

$$S_{iN}(x) = \sum_{k=0}^2 a_{ik} x^k, \text{ for } i = 1, 2,$$

where $a_{10} = f_1(0) = 0$ and $a_{20} = f_2(0) = 0$, then

$$S_{12}(x) = a_{11}x + a_{12}x^2$$

$$S_{22}(x) = a_{21}x + a_{22}x^2$$

By putting $S_{i2}(x), i=1,2$ in SNVIEK2 we get:

$$E_{12}(x, a_{1k}, k = \overline{0,2}) = x - x^2 + \int_0^x ((a_{11}t + a_{12}t^2) + (a_{21}t + a_{22}t^2)) dt - a_{11}x - a_{12}x^2$$

$$E_{22}(x, a_{2k}, k = \overline{0,2}) = x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \int_0^x ((a_{11}t + a_{12}t^2)^2 + (a_{21}t + a_{22}t^2)) dt - a_{21}x - a_{22}x^2 .$$

After substitute $E_{i2}(x, a_{1k}, k = \overline{0,2}), i=1, 2$ into equations (2.13), (2.16) and (2.19) and solving the system, we obtained require coefficients as:

$$a_{11} = a_{21} = 1, a_{12} = a_{22} = 0 .$$

Thus, the solution of the above system is

$$u_1(x) = a_{10} + a_{11}x + a_{12}x^2 = x \quad \text{and} \quad u_2(x) = a_{20} + a_{21}x + a_{22}x^2 = x ,$$

This is the exact solution.

Example 2: (Wazwaz, 2005)

Solve a system of non-linear VIEK2's:

$$u_1(x) = \sec(x) - x + \int_0^x ((u_1(t))^2 - (u_2(t))^2) dt$$

$$u_2(x) = 3 \tan(x) - x - \int_0^x ((u_1(t))^2 + (u_2(t))^2) dt$$

The exact solution of this system is:

$$u_1(x) = \sec(x) \quad \text{and} \quad u_2(x) = \tan(x)$$

Solution: Assume that the approximate solutions are:

$$S_{iN}(x) = \sum_{k=0}^N a_{ik} x^k, \quad i = 1, 2$$

where $a_{10} = f_1(0) = 1$ and $a_{20} = f_2(0) = 0$.

After solving this system by WRM's,

For $N=2$, we obtain the following approximate solutions of the system:

(i) Using Collocation Method (CM):

$$u_1(x) \approx 1 - \frac{40}{29987} x + \frac{836}{1623} x^2,$$

$$u_2(x) \approx \frac{736}{741} x + \frac{785}{7739} x^2.$$

(ii) Using Subdomain Method (SM):

$$u_1(x) \approx 1 - \frac{28}{53597} x + \frac{3428}{6735} x^2,$$

$$u_2(x) \approx \frac{1183}{1187} x + \frac{143}{1888} x^2.$$

(iii) Using Galerkin Method (GM):

$$u_1(x) \approx 1 - \frac{15}{2249} x + \frac{202}{361} x^2,$$

$$u_2(x) \approx \frac{1393}{1662} x + \frac{1243}{2295} x^2.$$

For $N=3$, we obtain the following approximate solutions of the system:

(i) Using Collocation Method (CM):

$$u_1(x) \approx 1 - \frac{52}{6599} x + \frac{179}{453} x^2 + \frac{383}{1284} x^3,$$

$$u_2(x) \approx \frac{449}{441} x - \frac{292}{2119} x^2 + \frac{767}{1331} x^3.$$

(ii) Using Subdomain Method (SM):

$$u_1(x) \approx 1 + \frac{43}{71278} x + \frac{6423}{13280} x^2 + \frac{442}{4037} x^3,$$

$$u_2(x) \approx \frac{1099}{1084}x - \frac{634}{5775}x^2 + \frac{1276}{2355}x^3.$$

(iii) Using Galerkin Method (GM):

$$u_1(x) \approx 1 + \frac{1}{25594}x + \frac{569}{1148}x^2 + \frac{677}{12281}x^3,$$

$$u_2(x) \approx \frac{650}{633}x - \frac{437}{2145}x^2 + \frac{309}{452}x^3.$$

Table (1): comparison between the least square errors of Example 2.

Methods	N=2		N=3	
	Least square errors for		Least square errors for	
	$u_1(x)$	$u_2(x)$	$u_1(x)$	$u_2(x)$
CM	8.0421×10^{-2}	1.5473×10^{-2}	5.0644×10^{-2}	3.5296×10^{-3}
SM	1.1720×10^{-1}	9.9768×10^{-3}	6.5657×10^{-2}	1.5468×10^{-3}
GM	1.6240×10^{-1}	3.7186×10^{-1}	1.1991×10^{-1}	5.0132×10^{-2}

Example 3: (Jumaa, 2005)

Solve a system of non-linear VIEK2's:

$$u_1(x) = \frac{1}{4} - \frac{1}{4}e^{2x} + \int_0^x (x-t)u_2^2(t)dt$$

$$u_2(x) = -xe^x + 2e^x - 1 + \int_0^x te^{-2u_1(t)} dt$$

The exact solution of this system is:

$$u_1(x) = -0.5x \quad \text{and} \quad u_2(x) = e^x$$

Solution: Assume that the approximate solutions are:

$$S_{iN}(x) = \sum_{k=0}^2 a_{ik}x^k, \quad i = 1, 2,$$

Where $a_{10} = f_1(0) = 0$ and $a_{20} = f_2(0) = 1$,

After solving this system by WRM's, we obtain the following solutions of the system:

(i) Using Collocation Method (CM):

$$u_1(x) \approx -\frac{555}{1138}x - \frac{91}{2792}x^2,$$

$$u_2(x) \approx 1 + \frac{1515}{1327}x + \frac{317}{3906}x^2.$$

(ii) Using Subdomain Method (SM):

$$u_1(x) \approx -\frac{321}{629}x + \frac{211}{3833}x^2,$$

$$u_2(x) \approx 1 + \frac{1409}{1464}x + \frac{2531}{6198}x^2.$$

(iii) Using Galerkin Method (GM):

$$u_1(x) \approx -\frac{1408}{2715}x + \frac{310}{3269}x^2,$$

$$u_2(x) \approx 1 + \frac{394}{373}x + \frac{677}{604}x^2.$$

Table (2): comparison between the least square errors of Example 3.

Methods	Least square errors for	
	$u_1(x)$	$u_2(x)$
CM	5.2255×10^{-6}	7.9235×10^{-4}
SM	3.0147×10^{-6}	6.8120×10^{-3}
GM	9.5006×10^{-6}	2.0700×10^{-1}

Conclusion

Non-linear system of Volterra integral equations of the second kind is usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions, for this reason the presented method has been proposed for approximate solution. In practice we conclude that we get the exact solution if $f_i(x)$, $i=1, 2, \dots, m$ is polynomial, also we get the good accuracy if we chose N sufficiently large.

References

- Atkinson, K. E., (1992): A Survey of Numerical Methods for Solving Nonlinear Integral Equations, J. of Integral Equations, Vol. 4, pp. 15-46.
- Babolian, E. and Biazar, J., (2000): Solution of a System of Nonlinear Volterra Integral Equations of the Second Kind, Far East J. Math. Sci. (FJMS), Vol. 2, pp. 935-946.
- Boyed, J. P., (2000): Chebyshev and Fourier Spectral Method, DOVER publications, Inc., Minneola, New York.
- Brunner, H., (1990): On the Numerical Solution of Nonlinear Volterra-Fredholm Integral Equations by Collocation Methods, SIAM J. NUMER. ANAL., Vol. 27, pp. 987-1000.
- Brunner, H., Pedas, A. and Vainikko, G., (1999): The Piecewise Polynomial Collocation Method for Nonlinear Weakly singular Volterra Equations, Mathematic of Computation, Vol. 68, pp. 1079-1095.

- Chambers, L. I. G., (1976): Integral Equations: A Short Course, International Text Book Company Limited.
- Chapra, S. C. and Canale, R. R., (2002): Numerical Method for Engineers: With Software and Programming Application, Fourth Edition, McGraw-Hill.
- Davis, P. J.,(1975): Interpolation and Approximation, Dover Publication, Inc. New York.
- Delves, L. M. and Walsh, J.,(1974): Numerical Solution of Integral Equations, Claredon Press Oxford.
- Hall, G. and Watt, J. M., (1976): Modern Numerical Methods for Ordinary Differential Equations, Oxford University Press.
- Jumaa, B. F., (2005): On Approximate Solutions to a system of Non-linear Volterra Integral Equations, D. Sc. Thesis, University of Technology, Department of Applied Science.
- Kuthen, J.-P. and Brunner, H., (1997): Continuous Collocation Approximations to Solutions of First Kind Volterra Equations, Mathematic of Computation, Vol. 66, pp. 1441-1459.
- Mahmoudi, Y., (2005): Wavelet Galerkin Method for Numerical Solution of Nonlinear Integral Equation, Applied Mathematic And Computation, pp. 1119-1129.
- Polyanin, A. D. and Manzhirov, A. V., (1998): Handbook of Integral Equations, CRC Press LLC.
- Saeed, R. K., (2006): Computational Methods for Solving System of Linear Volterra Integral and Integro-Differential Equations, PhD. Thesis, University of Salahaddin, Department of Applied Science.
- Wazwaz, A.-M., (2005): The Modified Decomposition Method for Analytic Treatment of Non-Linear Integral Equations and Systems of Non-Linear Integral Equations, International Journal of Computer Mathematics, Vol. 82, pp. 1107-1115.

الحلّ التقريبي لنظام معادلات فولتيرا التكاملية اللاخطية للنوع الثاني بالطرق المتبقية الموزونة

روستم كريم سعيد* جنار شهاب احمد**

* كلية العلوم – جامعة صلاح الدين/أربيل

** كلية العلوم – جامعة السليمانية

تاريخ الاستلام: ٢٠٠٧/٤/١١، تاريخ القبول: ٢٠٠٨/٤/٨

الخلاصة

في هذه الدراسة، ثلاثة أنواع من الطرق المتبقية الموزونة (طرق تجميع، مجال ثانوي و كالركين) المقدمّة لإيجاد الحلّ التقريبي (المضبوط أحياناً) للنظام الغير خطي لمعادلات فولتيرا التكاملية من النوع الثاني. وبيّنا كفاءة الطرق الموصوفة بحلّ بعض الأمثلة.