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Semi-Essential Submodules and Semi-Uniform Modules

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Abstract

In this work,we give generalizations for the concepts essential submodule and uniform module.We call an R-submodule N of M semi-essential if $N\cap P\neq 0$ for each nonzero prime R- submodule P of M, and we call an R- module M semi - uniform if every nonzero R-submodule N of M is semi-essential. Moreover, we generalize some properties of essential R-submodules to semi-essential R-submodules,and we generalize some properties of uniform R-modules to semi-uniform R-module. We also give conditions under them an R-submodule N of a multiplication R- module M becomes semi- essential. Furthermore, we give some conditions under them an R-module M satisfies ACC(DCC) on semi-essential R-submodules.

Introduction

Let R be a commutative ring with unity and let M be a unitary R-module. A nonzero R-submodule N of M is called essential if $N\cap L\neq 0$ for each nonzero R-submodule L of M and M is called uniform if every nonzero R-submodule N of M is essential(Kasch,1982).In section one, we introduce a semi-essential R-submodule concept as a generalization of essential R-submodule concept. Our main concerns in this section are to give a characterization for semi-essential submodules and generalize some known properties of essential R-submodules to semi-essential R- submodules. In section two, we give conditions under them an R-submodule N of a faithful multiplication R-module M becomes semi- essential (Th.2.1 and Th.2.2).In section three, we give some conditions under them an R-module M satisfy ACC (DCC) on semi- essential R- submodules (Prop.3.3 and Th.3.4).In section four, we present a semi-uniform module concept as a generalization of a uniform module concept. We also generalize a characterization and some properties of uniform modules to semi-uniform modules.

Semi-Essential Submodules

An R-submodule N of M is called essential if $N\cap L\neq 0$ for each nonzero R-submodule L of M. In this section, we give a generalization for essential submodule concept namely a semi-essential submodule,and we study some properties of semi-essential submodules.Recall that an R-submodule P of M is called prime if P is proper and whenever rx∈P for r∈Rand x∈M,then either x∈P or r∈ (P:M), where (P:M)={r∈ R: rM ⊆P}(Lu,1981).

Definition 1:

A nonzero R-submodule N of M is called semi-essential if N∩P≠ 0 for each nonzero prime R-submodule P of M.

Examples 2:

1- Every essential R-submodule is semi-essential. The converse is not true in general as the following example shows :consider Z_{12} as a Z-module. The Z-submodule ($\bar{6}$) is a semi-essential Z-submodule of Z_{12} ,but ($\bar{6}$) is not essential.
2- If M is a semi-simple R-module, then M is the only semi-essential R-submodule of M.

The following proposition gives a necessary and sufficient condition for an R-submodule to be a semi-essential.The proof is easy and hence is omitted.

Proposition 3:

A nonzero R-submodule N of M is semi-essential if and only if for each nonzero prime R-submodule P of M there exists x ∈ P and there exists r ∈ R such that 0≠rx ∈N.The proof of the following proposition is straightforward and hence is omitted.

Proposition 4:

Let M be an R-module and let N_1, N_2 be R-submodules of M such that N_1 is an R-submodule of N_2. If N1 is a semi-essential R-submodule of M, then N_2 is a semi-essential R-submodule of M.The converse of Prop.1.4 is not true in general. The following example indicates that.

Example 5:

Consider Z_{12} as a Z-module. ($\bar{4}$) is a semi-essential Z-submodule of ($\bar{2}$) and ($\bar{2}$) is a semi-essential Z-submodule of Z_{12}.But ($\bar{4}$)∩($\bar{3}$) =($\bar{0}$),and($\bar{3}$) is a prime Z-submodule of Z_{12}. Therefore ($\bar{4}$) is not a semi-essential Z-submodule of Z_{12}.

Corollary 6:

Let N_1 and N_2 are R- submodules of M. If N_1∩N_2 is a semi-essential R-submodule of M,then $N_1$and $N_2$are semi-essential.The converse of corollary 1.6 is not true in general as the following example shows.

Example 7:

Consider Z_{36} as a Z-module. The prime Z-submodules of Z_{36} are $(\bar{2})$ and$(\bar{3})$.Now, $(\overline{12})\cap(\bar{2})=(\overline{12})$ and $(\overline{12})\cap(\bar{3})=(\overline{12})$.Thus $(\overline{12})$ is semi-essential. Also $(\overline{18})\cap(\bar{2})=(\overline{18})$and $(\overline{18})\cap(\bar{3})=(\overline{18})$.Therefore$(\overline{18})$is semi-essential.But $(\overline{12})\cap(\overline{18})=(\bar{0})$which is not semi-essential.In the following proposition, we give a condition under which the converse of corollary 1.6 is true.

Proposition 8:

Let N_1 and N_2 are R-submodules of M such that N_1 is essential and N_2 is semi-essential. Then $N_1 \cap N_2$ is a semi-essential R-submodule of M.

Proof:

It is straightforward. Recall that the annihilator of an R-module M is defined as the following: Ann(M)={r∈ R: rx=0 for all x ∈M} Before we give the following proposition we need the following lemma.

Lemma 9:

Let N be an R-submodule of M and let P be a prime R-submodule of M. If (N∩P: x)=ann(M),for each x∈M and x∉N∩P, then N∩P is a prime R-submodule of M.

Proof:

Let rm∈N∩P,where r∈R and m∈M.Suppose that m∉N∩P.Since rm∈N∩P, then r∈(N∩P:m).It follows that r∈ann(M),and consequently r∈(N:M)∩(P:M). Thus r∈(N∩P:M)(Larsan,1971).Therefore N∩P is a prime R-submodule of M.The following proposition present another condition under which the converse of Corollary 1.6 is true.

Proposition 10:

let $N_1$and $N_2$are semi-essential R-submodules of M.If$(N_1\cap P:x)$=ann(M), for each prime R-submodule P of M,for each x∈M and x∉$N_1\cap P$,then $N_1\cap N_2$ is semi-essential.

Proof:

Let P be a nonzero prime R-submodule of M.By Lemma 1.9,$N_1\cap P$ is a prime R-submodule of M.Thus$(N_1\cap N2)\cap P=N_2\cap(N_1\cap P)\neq 0$.Therefore $N_1\cap N_2$ is semi-essential.

Definition 11:

Let M be an R-module and let N be an R-submodule of M. A prime R-submodule L of M is called semi-relative intersection complement of N in M if N∩P=0,where P is a prime R-submodule of M,such that L⊆ P,then L=P.

Proposition 12:

Let N be a nonzero R-submodule of M and let L is a nonzero prime R-submodule of M .Then L is a semi-relative intersection complement of N in M if and only if $(N\oplus L)/L$ is a semi-essential R-submodule of M/L.

Proof:

Let $g:M\rightarrow M/L$ be the natural map,and let L be semi-relative intersection complement of N in M.Let K be a nonzero prime R-submodule of M/L such that$((N\oplus L)/L)\cap K=o$.There exists a prime R-submodule P of M such that $P=g^{-1}(K)$ and $g(P)=K=P/L$.Thus$((N\oplus L)/L)\cap P/L=0$and hence$(N\oplus L)\cap P=L$.Therefore $N\cap P$ is an R-submodule of $N\cap L$.Since L is semi-relative intersection complement of N in M,then $N\cap L=0$.It follows that $N\cap P=0$.This implies that L=P,and consequently $K=P/L=0$.Therefore$(N\oplus L)/L$ is a semi-essential R-submodule of M/L.
Conversely;let$(N\oplus L)/L$ is a semi-essential R-submodule of M/L,and let P be a prime R-submodule of M such that $L\subseteq P$ and $N\cap P=0$. Suppose that $x\in(N\oplus L)\cap P$. Thus x=n+y=p, where $n\in N$, $y\in L$ and $p\in P$. This implies that $n=p-y\in N\cap P=0$,and hence n=0.Therefore $x=y\in L$,and consequently$(N\oplus L)\cap P=L$. It follows that$((N\oplus L)/L)\cap P/L=0$.But P/L is a prime R-submodule of M/L and$(N\oplus L)/L$ is a semi-essential R-submodule of M/L,so P/L=0 which implies that P=L.Therefore L is semi-relative intersection complement of N in M. The radical of an R-module M (denoted rad(M)) is the intersection of all prime R-submodules of M, i.e,$rad(M)=\bigcap_{p\in spec(M)} P$,where spec(M)={ P:P is a prime R-submodule of M}, unless no such primes exist, in which case rad(M)=M.

Proposition 13:

Let M and L be R-modules and let $f:M\rightarrow L$ be an R- epimorphism such that $ker(f)\subseteq rad(M)$.If N is a semi-essential R-submodule of L,then $f^{-1}(N)$is a semi-essential R-submodule of M.

Proof:

Suppose that $f^{-1}(N)\cap P=0$, where P is a prime R-submodule of M. Since $ker(f)\subseteq rad(M)\subseteq P$, for each prime R-submodule P of M, then f(P)is a prime R-submodule of L. This implies that $N\cap f(P)=0$. Since N is a semi-essential R-submodule of L,then f(P)=0.Thus $P\subseteq f^{-1}(0)=kerf\subseteq f^{-1}(N)$,and hence $f^{-1}(N)\cap P=0$. This means that P=0.Therefore,$f^{-1}(N)$is a semi-essential R-submodule of M.

Definition 14:

Let M and N be R-modules. An R-homomorphism $f: M\rightarrow N$ is called semi-essential if f(M) is a semi-essential R-submodule of N.The proof of the following proposition is easy and hence is omitted.

Proposition 15:

N is a semi-essential R-submodule of M if and only if the inclusion function i: N→M is a semi-essential R-monomorphism.

Semi-Essential Submodules in Multiplication Modules

An R-module M is called multiplication if every R-submodule N of M is of the form EM for some ideal E of R(Barnard,1988)and an R-module M is called faithful if ann(M)=0.In this section,we give a condition under them an R-submodule N of a faithful multiplication R-module M becomes semi-essential(Th.2.1 and Th.2.2).We preface the section by the following result.

Theorem 1:

Let M be a faithful multiplication R-module and N is an R-submodule of M such that N=EM for some ideal E of R. Then N is semi-essential if and only if E is semi-essential.

Proof:

Assume that N is semi-essential and E∩B=0,where B is a prime ideal of R. Since M is a faithful multiplication R-module,then (E∩B)M=EM∩BM=0. Now,BM is a prime R-submodule of M (El-Baste,1988) and N=EM is a semi-essential R-submodule of M, so BM=0. It follows that B=0. Therefore E is a semi-essential ideal of R.
Conversely; let N∩P=0,where P is a nonzero prime R-submodule of M. Since M is multiplication, then there exists a prime ideal B of R such that P=BM (El-Baste, 1988). Hence N∩P = EM∩BM = (E∩B)M=0. But M is faithful, so E∩B=0. Since E is a semi-essential ideal of R, then B=0. Therefore P=BM=0, and consequently N is a semi-essential R-submodule of M. We also give in the following theorem a necessary and sufficient condition for an R-submodule N of M to be semi-essential.

Theorem 2:

Let M be a faithful multiplication R-module.Then N is a semi-essential R-submodule of M if and only if(N:x)is a semi-essential ideal of R for each x∈M.

Proof:

Suppose that N is semi-essential.Since M is a faithful multiplication R-module,then(N:M)is a semi-essential ideal of R(Th.2.1).But (N:M)⊆(N:x) for each x∈M,so N=(N:M)M⊆(N:x)M (El-Baste,1988).Hence(N:x)M is a semi-essential R-submodule of M (Prop.1.4), and consequently (N:x) is a semi-essential ideal of R (Th.2.1).

Conversely,assume that (N:x) is a semi-essential ideal of R for each x∈M. Let P be a nonzero prime R-submodule of M and let 0≠y∈P.Thus(N:y)is semi-essential. Since M is multiplication, then P=BM, where B is a prime ideal of R(El-Baste,1988).Hence(N:y)∩B≠0.By assumption M is faithful,so (N:x)M∩BM≠0.Thus N∩P≠0, and consequently N is a semi-essential R-submodule of M.A nonzero prime R-submodule N of M is called minimal prime if there exists a prime R-submodule P of M such that P⊆N,then P=N (El-Baste,1988).The following proposition shows that under certain condition a prime R-submodule of a faithful multiplication R-module becomes semi-essential.

Proposition 3:

Let M be a faithful multiplication R-module and let N be a nonzero prime R-submodule of M. If N is not minimal prime, then N is semi-essential.

Proof:

Since M is multiplication and N is prime, then there exists a prime ideal B of R such that ann(M)⊆B and N=BM (El-Baste,1988).Let P a nonzero prime R-submodule of M such that N∩P=0. Since N is not minimal prime, then there exists a minimal prime R-submodule L of M such that L⊂N (Ahmed,1992).Thus there exists a minimal prime ideal A of R such that ann(M)⊆A and L=AM≠M (Ahmed,1992).Now,(B∩(P:M))M=BM∩(P:M)M= N∩P=0.But M is faithful, so B∩(P:M)=0.Therefore B∩(P:M)⊆A and consequently either B⊆A or (P:M)⊆A. If B⊆A, then BM⊆ AM. Whence N⊆L, a contradiction. If (P:M)⊆A, then (P:M)M⊆AM. It follows that P⊆L⊆N.Hence 0=N∩P=P which is a contradiction.This prove that N∩P≠0 and consequently N is semi-essential.

Modules with ACC and DCC on Semi-Essential Submodules

An R-module M is said to be satisfy the ascending chain condition (abbreviated ACC) if each ascending chain of R-submodules of M terminates. Moreover, M is called Noetherian R-module if and only if M satisfies ACC, and M is said to be satisfy the descending chain condition (abbreviated DCC) on R-submodules if each descending chain of R-submodules of M terminates (Larsan,1971) ,(Naoum,2004).In this section, we try to answer the following question when does an R-module M satisfy ACC(DCC) on semi-essential R-submodules?. We give some conditions under them an R-module M satisfies ACC(DCC) on semi-essential R-

Submodules (Prop.3.3).We also prove that a finitely generated faithful multiplication R-module M satisfies ACC (DCC) on semi-essential R-submodules if and only if R satisfies ACC (DCC) on semi-essential ideals of R (Th.3.4).We start by the following definition.

Definition 1:

An R-module M is called satisfied the a scending chain condition on semi-essential R-submodules if each ascending chain of semi-essential R-submodules $N_1 \subseteq N_2 \subseteq ... \subseteq N_n \subseteq$...terminates.The proof of the following proposition is routine and hence is omitted.

Proposition 2:

Let M be an R-module and let N be an R-submodule of M such that $N \subseteq rad(M)$. If M satisfies ACC(DCC) on semi-essential R-submodules, then M/N satisfies ACC(DCC) on semi-essential R-submodules.

Proposition 3:

An R-module M satisfies ACC on semi-essential R-submodules if each semi-essential R-submodule of M is finitely generated.

Proof:

Let $N_1 \subseteq N_2 \subseteq ... \subseteq N_n \subseteq ...$ be an ascending chain of semi-essential R-submodules of M. Put $\sum_{i \in I} N_i = N$.Thus N is a semi-essential R-submodule of M (Prop.1.4),and hence N is finitely generated. Therefore there exists a finite set $I_0 \subseteq I$ such that $N = \sum_{i \in I} N_i$ .Hence the chain terminates.The following theorem gives the relation between the multiplication R-module M which satisfies ACC on semi-essential R-submodules and the ring R which satisfies ACC on semi-essential ideals.

Theorem 4:

Let M be a finitely generated faithful multiplication R-module. Then M satisfies ACC(DCC) on semi-essential R-submodules if and only if R satisfies ACC(DCC) on semi-essential ideals.

Proof:

Let $E_1 \subseteq E_2 \subseteq ... \subseteq E_n \subseteq$...be an ascending chain of semi-essential ideals of R.Then $E_1M \subseteq E_2M \subseteq ... \subseteq E_nM \subseteq$...is an ascending chain of semi-essential R-submodules of M (Th.2.1).Since M satisfies ACC on semi- essential R-submodules,then there exists a positive integer n such that $E_nM = E_{n+1}M = ...$. But M is a finitely generated faithful multiplication R-module,then $E_n = E_{n+1} = ...$.

(El-Baste,1988).Hence R satisfies ACC on semi-essential ideals.
Conversely; let $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$be an ascending chain of semi-essential R-submodules of M.Since M is multiplication,then $N_i=E_i M$ for some semi-essential ideals E_i of R,for each i=1,2,3,...,n,...(Th.2.1).Thus $E_1M \subseteq E_2M \subseteq \ldots \subseteq E_nM \subseteq \ldots$and since M is afinitely generated faithful multiplication R-module,then $E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n \subseteq \ldots$ is an ascending chain of semi-essential ideals of R(El-Baste,1988).But R satisfies ACC on semi-essential ideals, thus there exists a positive integer n such that $E_n=E_{n+1}=\ldots$ .Hence $E_nM= E_{n+1}M=\ldots$ .Therefore M satisfies ACC on semi-essential R-submodules.

Theorem 5:

Let M be a finitely generated faithful multiplication R-module, then the following statements are equivalent.

1-M satisfies ACC(DCC) on semi-essential R-submodules.
2-R satisfies ACC(DCC) on semi-essential ideals.
3-S=End(M) satisfies ACC(DCC) on semi-essential ideals.
4-M satisfies ACC(DCC) on semi-essential R-submodules as an S-module.

Proof:

(1)$\Leftrightarrow$(2) By Th.3.4.
(2)$\Leftrightarrow$(3) Since M is a finitely generated faithful multiplication R-module, then $R \cong S$ (Naoum,1994). Thus R satisfies ACC on semi-essential ideals if and only if S satisfies ACC on semi-essential ideals.
(3)$\Leftrightarrow$(4) By Th.3.4 and $R \cong S$.
(1)$\Leftrightarrow$(4) By (Naoum,1994),$R \cong S$. Therefore M satisfies ACC on semi-essential R-submodules as an S-module.

Semi-Uniform Modules:

Recall that a nonzero R-module M is called uniform if every nonzero R-submodule of M is essential(Goodearl,1972).In this section,we give a semi-uniform module concept as a generalization of uniform module concept. We also generalize some properties of uniform modules to semi-uniform modules.

Definition 1:

A nonzero R-module M is called semi-uniform if every nonzero R-submodule of M is semi-essential. A ring R is called semi- uniform if R is a semi-uniform R-module.

Examples 2:

1-Each uniform R-module is semi-uniform, but the converse is not true in general as the following example indicts that.

Consider Z_{36} as a Z-module. Z_{36} is semi-uniform. But Z_{36} is not uniform, since $(\overline{18})\cap(\overline{12})=(\overline{0})$.

2-Each simple R-module is semi-uniform.

It is well-known the uniform property is hereditary,but the semi-uniform property is not hereditary. Consider the following example.

Example 3:

Z_{36} is a semi-uniform Z-module.$(\overline{3})$ is a Z-submodule of Z_{36}.We claim that$(\overline{3})$is not a semi-uniform Z-module.The prime Z-submodules of $(\overline{3})$are $(\overline{6})$and$(\overline{9})$.$(\overline{12})$ is a Z-submodule of $(\overline{3})$and $(\overline{12})\cap(\overline{9})=(\overline{0})$.Thus $(\overline{3})$ is not a semi-uniform Z-module.It is Known that the intersection of uniform R-module with any R-module is uniforme. But this property does not hold in case the module is semi-essential.The following example shows that.

Example 4:

Z_{36} is a semi-uniform Z-module.$(\overline{6})$ is not a semi-uniform Z-module. $Z_{36}\cap(\overline{6})=(\overline{6})$ is not semi-uniform.

Theorem 5:

Let M be a faithful multiplication R-module. Then M is a semi-uniform R-module if and only if R is a semi-uniform ring.

Proof:

Suppose that M is semi-uniform and let A be a nonzero ideal of R.Thus AM is a semi-essential R-submodule of M. By Th.2.1, A is a semi-essential ideal of R.

Conversely, assume that R is semi-uniform and N is an R-submodule of M. Since M is multiplication, then there exists an ideal B of R such that N=BM. But R is semi-uniform, so B is semi-essential. By Th.2.1, N is semi-essential. Recall that an R-module M is called torsionless if $\bigcap_{f\in M^*}$ ker(f)=0, where M^* =Hom(M, R) (Kasch,1982).

Proposition 6:

Let M be a faithful multiplication R-module and let for each f$\in M^*$,f is onto and ker(f)$\subseteq$rad(M).If T(M) is a semi-uniform ideal of R,then M is a semi-uniform R-module.

Proof:

Let N be a nonzero R-submodule of M and P is a nonzero prime R-submodule of M such that N$\cap$P=0. Since M is a faithful multiplication R-module,then (N:M)M$\cap$(P:M)M=0.Hence(N:M)$\cap$(P:M)=0.By(Kasch,1982), M is torsionless and consequently $\bigcap_{f\in M^*}$ ker(f)=0. It follows that there are f,g∈M^* such that f(N)≠0 and g(P)≠0. In fact, if f(N)=0 for all f∈M^*, then N⊆ $\bigcap_{f\in M^*}$ ker(f)=0, a contradiction. Similarly g(P)≠0.(N:M)⊇(N:M)f(M)= f((N:M)M)=f(N)⊆T(M)and(P:M)⊇(P:M) g(M)=g((P:M)M)=g=(P)⊆T(M). Then f(N)∩g(P) ⊆(N:M)∩(P:M) =(N∩P:M) = (0:M)= ann(M)=0 . Hence f(N)∩g(P)=0.This is a contradiction,since g(P) is a prime R-submodule of T(M).Therefore N∩P ≠ 0 which means that N is semi-essential.Hence M is semi-uniform.

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المقاسات الجزئية شبه الجوهرية و المقاسات شبه المنتظمة

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الخلاصة

في عملنا هذا قدمناتعميم لمفهوم المقاس الجزئي الجوهري ومفهوم المقاس المنتظم.حيث عرفنا المقاس الجزئي شبة الجوهري N من المقاس M بأنه مقاس جزئي غير صفري و $N \cap P \neq 0$ لكل مقاس جزئي أولي غير صفري P من M. وعرفنا مفهوم مقاس شبه المنتظمMبأنه مقاس يكون فيه كل مقاس غير صفري شبه جوهري. ثم عممنا بعض صفات المقاسات الجزئية الجوهرية إلى المقاسات الجزئية شبه الجوهرية. فضلا عن ذلك قدمنا تعميم لبعض خصائص المقاسات المنتظمة إلى المقاسات شبه المنتظمة. ثم أعطينا بعض الشروط و التي بموجبها يكون أي مقاس جزئي من مقاس جدائي شبه جوهري. و أخيرا درسنا المقاسات التي تحقق خاصيتي السلسلة(ACC(DCC على المقاسات الجزئية شبه الجوهرية.