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Resolvent Kernel and Haar Wavelet Techniques for Solving Coupled of Fredholm Integro-Differential Equations

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Abstract

This study focuses on the Fredholm integro-differential equations and is frequently found in areas of applied mathematics, physics, and engineering fields. To address these systems and obtain exact of solutions under suitable conditions, we propose a novel analytical approach. We choose the Haar wavelet colocation method for processing due to its simplicity, effectiveness, and ability to handle non-smooth solutions. The integral terms in these equations are determined using the trapezoidal rule, which effectively strikes a balance between accuracy and computational efficiency. These results are compared with analytical solutions. The comparisons show that the suggested strategy yields highly accurate results and offers a solid framework for solving the Fredholm integro-differential equations.

1. Introduction:

Systems of Integro-differential equation de Fredholm (SFIDEs) are widely used in various scientific and practical fields, with notable applications in glassmaking [1], hydrodynamics at the nanoscale [2], simulating the competition between the immune system and cancer cells [3], and studying the noise related phenomenon related [4]. Numerous studies have been conducted in recent years as a result of the SFIDEs' popularity. There are several techniques, such as the variational iteration technique [5], the Chebyshev polynomial technique [6], the Tau technique [7], the differential transformation technique [8], the Runge-Kutta techniques [9], the bloque impulse function technique [10], or even the spectral technique [11]. The Laguerre strategy for solving SFIDEs was examined by Zafer Elahi et al. in [12], in addition, numerical analysys have extensively investigated methods for solving integral equations of both linear and nonlinear types. Notably, the works of Saber, Surme R. and colleagues, as well as Najem et al., focus on systems involving Fredholm integro-differential equations [13], [14]. Additional contributions to this area are presented in ref.[15, 16, 17]. The noyau de Resolvent has been studied by a variety of researchers in [18, 19, 20, 21]. The Resolvent kernels technique is generally regarded as an effective method for solving the Fredholm integro-differential equations. Using the properties of the Resolvent kernels, this technique provides a systematic way to transform integro-differential equations into more efficient equivalent structures. Lepik [22], Babolian [23], and Aziz et al. have employed the Haar vague Lette technique. [24] to discuss the Fredholm-Nolinear integrals. The Haar wavelet approach was recently used by Najem A. Mohammad and associates [25, 26, 27] have worked to solve both integral and partial integro-differential equations. Haar wavelets are advantageous due to their simplicity, orthogonality, and compact support. The main advantages include sparse representation, quick transformations, and the ability to create effective matrix representation techniques. Since the base Haar is the most straightforward type of wavelet spline, with a polynomial degree equal to zero, it offers a relatively low computational cost. Therefore, Haar wavelets are frequently used to solve a variety of problems by converting them into linear equation systems with collocation points, which are then solved using MATLAB, as explained in [25].

In this study, we presented both analytical and numerical methods to solve the SFIDEs. [4]

$$\sum_{i=1}^{n}\alpha_i\phi_1^{(n)}(s)=\psi_1(s)+\lambda_1\int_a^b[k_{11}(s,r)\phi_1(r)+k_{12}(s,r)\phi_2(r)]dr$$
$$\sum_{j=1}^{m}\beta_j\phi_2^{(n)}(s)=\psi_2(s)+\lambda_2\int_a^b[k_{21}(s,r)\phi_1(r)+k_{22}(s,r)\phi_2(r)]dr, \tag{1}$$

with initial conditions:
$\phi_1^{(i-1)}(0)=\alpha_i,\qquad \phi_2^{(j-1)}(0)=\beta_j$
where $i,j=1,2,3,\ldots,n,$ and $\phi_1^{(n)}(s)=(d^n\phi_1)/(ds^n)$, $\phi_2^{(n)}(s)=(d^n\phi_2)/(ds^n)$, denote the nth derivatives of ϕ_1 and ϕ_2 with respect to s, respectively (Wazwaz, 2011).

The functions $\psi_1(s),\psi_2(s),k_{11}(s,r),k_{12}(s,r),k_{21}(s,r)$ and $k_{22}(s,r)$ are assumed to be known and sufficiently smooth, real-valued functions defined on the domain$[a,b]\times[a,b]$. The parameters λ_1 and λ_2 are given real parameters, involving complex kernels or high nonlinearity.

Although the proposed kernel solver methodology aims provides a solid framework for analysis, efforts aim to enhance high computational precision and efficiency, on-going particularly in equations with complex kernels or high nonlinearity. These components how to importance of ralldating and retiming. This highlight structure at this work is outlined as followed. Using the technique of the Resolvent kernels, Section 2 suggests an innovative approach to solving a specific case of the Fredholm linear integro-differential equations n systems (SLFIDEs). Section 3 discusses the Haar wavelet technique via collocation (HWCM), while Section 4 presents a numerical model that uses Haar wavelet to solve a system of second-order (FIDE). The effectiveness and precision of the techniques presented in Sections 2 and 4 are examined in section 5 by applying them to various test examples. Section 6 offers a detailed analysis of these cases by comparing the numerical and exact results of a first-order system (FIDE) that uses both the wavelet Haar and the Resolvent kernels methods. Additionally, it examines the specific case of the SLFIDEs addressed in Section 2 using the Resolvent kernels technique. Finally, Section 7 sums up the study's main results and conclusions. This section's goal is to present analytical and approximate methods for

2. Description Methods:

dealing with coupled systems (FIDE). The techniques mentioned include the use of Haar wavelets and the Resolvent kernels approach, which offer precise and effective solutions to these complex systems.

2.1 Analytical Method:

We introduce a novel technique to solve a special case of (SFIDEs) in Eq. (1), given as follows:

$$\phi_1^{(n)}(s)=\psi_1(s)+\lambda_1\int_a^b[k_{11}(s,r)\phi_1(r)+k_{12}(s,r)\phi_2(r)]dr,$$
$$\phi_2^{(n)}(s)=\psi_2(s)+\lambda_2\int_a^b[k_{21}(s,r)\phi_1(r)+k_{22}(s,r)\phi_2(r)]dr. \tag{2}$$

After introducing the problem through a system of integral Fredholm linear equations (SLFIE), the system is then solved by combining the direct calculation method with the noyaure solved technique.

Now, by integrating both sides of system (2), n times from 0 to s until we obtain $\phi_1(s)$ and ϕ_2 (s):

$$\phi_1(s)=w_1(s)+\lambda_1\int_a^b[g_{11}(s,r)\phi_1(r)+g_{12}(s,r)\phi_2(r)]dr,$$
$$\phi_2(s)=w_2(s)+\lambda_2\int_a^b[g_{21}(s,r)\phi_1(r)+g_{22}(s,r)\phi_2(r)]dr. \tag{3}$$

Where $w_1(s)$ is the result of integrating $\psi_1(s)$ with those terms at initial conditions and $w_2(s)$ is the result of integrating $\psi_2(s)$ with those terms at initial conditions,
$g_{11}(s,r),g_{12}(s,r),g_{21}(s,r),g_{22}(s,r)$
are the results of integrating number of thekernels.
The kernels:
$k_{11}(s,r),k_{12}(s,r),k_{21}(s,r),k_{22}(s,r)$.
The new kernels $g_{12}(s,r)$ and $g_{21}(s,r)$ are separable functions, defined as follows:

$$g_{12}(s,r)=d_1(s)h_1(r),$$
$$g_{21}(s,r)=d_2(s)h_2(r). \tag{4}$$

Substituting Eq.(4) into Eq.(3), we obtain:

$$\phi_1(s)=w_1(s)+\lambda_1\int_a^b g_{11}(s,r)\phi_1(r)dr+\lambda_1d_1(s)\int_a^b h_1(r)\phi_2(r)dr,$$
$$\phi_2(s)=w_2(s)+\lambda_2d_2(s)\int_a^b h_2(r)\phi_1(r)dr+\lambda_2\int_a^b g_{22}(s,r)\phi_2(r)dr. \tag{5}$$

To simplify Eq.(5), we define:

$$\begin{cases} A = \int_a^b h_1(r)\phi_2(r)dr \\ B = \int_a^b h_2(r)\phi_1(r)dr \end{cases} \quad (6)$$

and substituting Eq.(6) into Eq.(5), we obtain:

$$\begin{cases} \phi_1(s) = w_1(s) + \lambda_1 A d_1(s) + \lambda_1 \int_a^b g_{11}(s,r)\phi_1(r)dr, \\ \phi_2(s) = w_2(s) + \lambda_2 B d_2(s) + \lambda_2 \int_a^b g_{22}(s,r)\phi_2(r)dr \end{cases} \quad (7)$$

Now we solve Eq.(7) using the Resolvent kernel method.
Let $K(s,r) = g_{11}(s,r)$ and $K^*(s,r) = g_{22}(s,r)$,
and $K_1(s,r) = K(s,r)$, then the iterated kernels are defined by:
$K_n(s,r) = \int_a^b K(s,z)k_{(n-1)}(z,r)$ dz,where $n \geq 2$,
and

$$R(s,r;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(s,r). \quad (8)$$

Using this representation, the solution of the general equation given in Eq. (7) takes the form:

$$\phi(s) = \psi(s) + \lambda \int_a^b R(s,r;\lambda)\psi(r)dr. \quad (9)$$

By substituting Eq. (9) and inserting (8) into Eq.(7) , we derive the following expressions for $\phi_1(s)$ and ϕ_2 (s):

$$\phi_1(s) = w_1(s) + \lambda_1 A d_1(s) + \lambda_1 \int_a^b R(s,r;\lambda)(w_1(r) + \lambda_1 A d_1(r))dr,$$
$$\phi_2(s) = w_2(s) + \lambda_2 B d_2(s) + \lambda_2 \int_a^b R(s,r;\lambda)(w_2(r) + \lambda_2 B d_2(r))dr. \quad (10)$$

Using the result from Eq.(10) into Eq.(6) we find the value of A and B, then by substituting these values back into Eq. (10), we get the exact solution$(\phi_1(s), \phi_2(s))$ for the given system.

2.2 Haar Wavelet (HWCM):

$$H(s) = \begin{cases} 1 & for \quad s \in [\Upsilon_1, \Upsilon_2), \\ -1 & for \quad s \in [\Upsilon_2, \Upsilon_3), \\ 0 & otherwise, \end{cases} \quad (11)$$

Υ, where $\rho = 1,2,3.$ is determined as follows:

$\Upsilon_1 = a + (b-a)k/m,$
$\Upsilon_2 = a + ((b-a)(k+0.5))/m,$
$\Upsilon_3 = a + ((b-a)(k+1))/m$

The formula $i = m + k + 1$ is used to obtain the value of the index i,
and

$$s_q = a + \frac{(b-a)(q-0.5)}{N}, q = 1,2,...,N. \quad (12)$$

is used to find the collocation points

$$p_{i,1}(s) = \begin{cases} s - \Upsilon_1 & for \quad s \in [\Upsilon_1, \Upsilon_2) \\ \Upsilon_3 - s & for \quad x \in [\Upsilon_2, \Upsilon_3) \\ 0 & Otherwise \end{cases} \quad (13)$$

$$p_{i,2}(s) = \begin{cases} \frac{(s-\Upsilon_1)^2}{2} & for \quad x \in [\Upsilon_1, \Upsilon_2) \\ \frac{1}{4m^2} / \frac{(\Upsilon_3 - s)^2}{2} & for \quad x \in [\Upsilon_2, \Upsilon_3) \\ 0 & O.W, \end{cases} \quad (14)$$

In general, we have

$$p_{i,x+1}(s) = \begin{cases} \frac{1}{x!}(s-\Upsilon)^x & if \quad s \in [\Upsilon_1, \Upsilon_2) \\ \frac{1}{x!}\{(s-\Upsilon_1)^x - 2(s-\Upsilon)^x\} & if \quad s \in [\Upsilon_2, \Upsilon_3) \\ \frac{1}{x!}\{(s-\Upsilon_1)^x - 2(s-\Upsilon)^x + (s-\Upsilon)^x\} & if \quad s \in [\Upsilon_3, \Upsilon_1) \\ 0 & else \quad where \end{cases} \quad (15)$$

where $x = 0,1,2,3,...,N$. The value of $p_{i,1}(s)$ increases on $[\Upsilon_1(i), \Upsilon_2(i))$, decreases on $[\Upsilon_2(i), \Upsilon_3(i))$, and attains its maximum at $\Upsilon_2(i)$, Consequently,

$$\max_s \left(p_{i,1}(s)\right) = p_{i,1}(a_2(i)) = \frac{b-a}{2m} = \frac{b-a}{2^{j+1}} \quad (16)$$

and

$$\max_s (p_{i,1}(s)) = \frac{b-a}{4m^2} = \left(\frac{b-a}{2^{j+1}}\right)^2. \quad (17)$$

3. Development of the Numerical Method:

This section focuses on developing a numerical method that uses the Haar wavelet colocation technique to solve a specific variation of equation (1), this variation further and modelled as a system of different equations is reformulated as a system of higher-order functional integro-differential equations

$$c_3\phi_1''(s) + c_2\phi_1'(s) + c_1\phi_1(s)$$
$$= \psi_1(s) + \lambda_1 \int_a^b [k_{11}(s,r)\phi_1(r) + k_{12}(s,r)\phi_2(r)]dr,$$
$$b_3\phi_2''(s) + b_2\phi_2'(s) + b_1\phi_2(s) =$$
$$\psi_2(s) + \lambda_2 \int_a^b [k_{21}(s,r)\phi_1(r) + k_{22}(s,r)\phi_2(r)]dr \quad (18)$$

The initial conditions are:
$\phi_1^{'}(0)=\alpha_1, \phi_1(0)=\beta_1, \phi_2^{'}(0)=\alpha_2 \quad and \quad \phi_2(0)=\beta_2$
Here, the quantities $c_1,c_2,c_3,b_1,b_2,b_3,\alpha_1,\beta_1,\alpha_2$ and β_2 are constant. The functions $\psi_1(s),\psi_2(s)$ are known, λ_1 and λ_2 are given parameters and the kernels, $k_{11}(s,r),k_{12}(s,r),k_{21}(s,r)$ and $k_{22}(s,r)$ are defined. Assume:

$$\phi_1^{''}(s)=\sum_{j=1}^{2M} a_i h_i(s). \tag{19}$$

By integrating both side of Eq. 19 from 0 to s and using the initial conditions specified in Eq. 18, we obtain

$$\phi_1^{'}(s)=\alpha_1+\sum_{j=1}^{2M} a_i p_{i,1}(s) \tag{20}$$

By integrating both side of Eq. (20) from 0 to s and applying the initial conditions specified in Eq. (18), we get

$$\phi_1(s)=\beta_1+\alpha_1 s+\sum_{j=1}^{2M} a_i p_{i,2}(s) \tag{21}$$

By replacing s with r in Eq. (21) we obtain:

$$\phi_1(r)=\beta_1+\alpha_1 r+\sum_{j=1}^{2M} a_i p_{i,2}(r). \tag{22}$$

Using a similar approach, we can express the second derivative of $\phi_2(s)$ in terms of HWC as shown below:

$$\phi_1(s)=\beta_1+\alpha_1 s+\sum_{j=1}^{2M} a_i p_{i,2}(s) \tag{23}$$

By replacing s with r in Eq. (23) we obtain:

$$\phi_2(r)=\beta_2+\alpha_2 r+\sum_{j=1}^{2M} d_i p_{i,2}(r). \tag{24}$$

By inserting Eqs.(19)-(24) into the system defined in (18), we obtain the following system of equations:

$$\begin{aligned}
&c_3(\sum_{j=1}^{2M} a_i h_i(s))+c_2(\alpha_1+\sum_{j=1}^{2M} a_i p_{i,1}(s))+c_1(\beta_1+\alpha_1 s+\sum_{j=1}^{2M} a_i p_{i,2}(s))\\
&\quad=\psi_1(s)+\lambda_1\int_a^b[k_{11}(s,r)(\beta_1+\alpha_1 r+\sum_{j=1}^{2M} a_i p_{i,2}(r))+k_{12}(s,r)(\beta_2+\alpha_2 r+\sum_{j=1}^{2M} d_i p_{i,2}(r))]dr\\
&b_3(\sum_{j=1}^{2M} d_i h_i(s))+b_2(\alpha_2+\sum_{j=1}^{2M} d_i p_{i,1}(s))+b_1(\beta_2+\alpha_2 s+\sum_{j=1}^{2M} d_i p_{i,2}(s))\\
&\quad=\psi_2(s)+\lambda_2\int_a^b[k_{21}(s,r)(\beta_1+\alpha_1 r+\sum_{j=1}^{2M} a_i p_{i,2}(r))+k_{22}(s,r)(\beta_2+\alpha_2 r+\sum_{j=1}^{2M} d_i p_{i,2}(r))]dr.
\end{aligned} \tag{25}$$

The integral terms in Eq.(25) are discretized using the following formula
$\int_a^b f(x)dx \approx \frac{b-a}{2m}\sum_{m=1}^{2M} f(x_m)$
Using the collocation points:
$x_m=c_2+\frac{(c_1-S)(m-0.5)}{N}$

$$\begin{aligned}
&c_3(\sum_{j=1}^{2M} a_i h_i(s))+c_2(\alpha_1+\sum_{j=1}^{2M} a_i p_{i,1}(s))+c_1(\beta_1+\alpha_1 s+\sum_{j=1}^{2M} a_i p_{i,2}(s))\\
&\quad=\psi_1(s)+\lambda_1\frac{b-a}{2M}\sum_{m=1}^{2M}\begin{pmatrix} k_{11}(s,r_m)(\beta_1+\alpha_1 r_m+\sum_{j=1}^{2M} a_i p_{i,2}(r_m))\\ +k_{12}(s,r_m)(\beta_2+\alpha_2 r_m+\sum_{j=1}^{2M} d_i p_{i,2}(r_m))\end{pmatrix}\\
&b_3(\sum_{j=1}^{2M} d_i h_i(s))+b_2(\alpha_2+\sum_{j=1}^{2M} d_i p_{i,1}(s))+b_1(\beta_2+\alpha_2 s+\sum_{j=1}^{2M} d_i p_{i,2}(s))\\
&\quad=\psi_2(s)+\lambda_2\frac{b-a}{2M}\sum_{m=1}^{2M}\begin{pmatrix} k_{21}(s,r_m)(\beta_1+\alpha_1 r_m+\sum_{j=1}^{2M} a_i p_{i,2}(r_m))\\ +k_{22}(s,r_m)(\beta_2+\alpha_2 r_m+\sum_{j=1}^{2M} d_i p_{i,2}(r_m))\end{pmatrix}
\end{aligned} \tag{26}$$

By solving the system in Eq.(26), we obtain the Haar coefficients a_j and d_j. Then, by substituting these coefficients into Eqs.(21), and (23), we obtain the approximate solution to Eq.(18).

Theorem 4.1: Consider the function $f(s)=\frac{d^n\psi(s)}{ds^n}\varepsilon L^2(R)$, which is continuous on the interval [0,1], and assume that it's first derivative is bounded. That is, for all $s\varepsilon[0,1]$, there exists a constant ζsuch that $\left|\frac{df(s)}{ds}\right|\leq\zeta, n\geq 2$. Under these assumptions, the Haar wavelet method is convergent, meaning the error $\mid\varepsilon\mid$ tends to zero as $J\to\infty$.The convergence is of order two, and the error satisfies the following asymptotic estimate: $\parallel\varepsilon\parallel_2=O(\frac{1}{2^{2J+1}})$,
where
$\varepsilon=\mid\phi^{Exact}-\phi^{Approx.}\mid$.

Proof. A detailed explanation of the proof can be found in [28].

4. Computations:

The accuracy and validity of the proposed methods for solving the SFIDE, as described in sections two and four, are investigated using various examples in this section.

4.1 Analytical Solutions:

This section illustrates the application of the methods developed in Section 2 through a variety of examples.

Example 4.1: Consider the coupled systems of Fredholm integro-differential equations

$$p_{i,1}(s)=\begin{cases}\phi_1^{'}(s)=\pi\cos(\pi s)-\frac{\pi+3}{3\pi}+\int_0^1[r\phi_1(r)+\phi_2(r)]dr\\ \phi_2^{'}(s)=\frac{-2}{\pi}+\int_0^1[\phi_1(r)+6s\phi_2(r)]dr\\ \phi_1(0)=0 \quad and \quad \phi_2(0)=0.\end{cases} \tag{27}$$

By applying the Resolvent kernel method presented in Section 2 to Example 4.1, and integrating Eq. (27) from 0 to s, we obtain the following system of integral equations:

$$\begin{aligned}\phi_1(s)&=\sin(\pi s)-\tfrac{\pi+3}{3\pi}s+s\int_0^1[r\phi_1(r)+\phi_2(r)]dr\\ \phi_2(s)&=\tfrac{-2}{\pi}s+\int_0^1[s\phi_1(r)+3s^2\phi_2(r)]dr\end{aligned} \tag{28}$$

Let:

$$A=\int_0^1\phi_2(r)dr \tag{29}$$

Substituting equation (29) into (28) then, we get

$$\phi_1(s)=\sin(\pi s)-\frac{\pi+3}{3\pi}s+As+\int_0^1 sr \quad \phi_1(r)dr. \tag{30}$$

Rewriting Eq.(30) in the standard form:

$\phi_2(s)=\psi(s)+\lambda\int_a^b k(s,r)\phi_2(r)dr$,
where:
$\psi(s)=\sin(\pi s)-\frac{\pi+3}{3\pi}s+As, \lambda=1, k(s,r)=sr \quad and \quad a=0, b=1$
The iterated kernels $k_n(s,r)$ is defined as:

$$k_n(s,r)=\begin{cases} sr & n=1,\\ \int_0^1 k(s,z)k_{n-1}(z,r)dz=(\frac{1}{3})^{n-1}sr & n\geq 2,\end{cases} \tag{31}$$

and

$$(s,r;\lambda)=\frac{3}{2}sr. \tag{32}$$

Substituting Eq. (32) is inserted into the following expression allows as to compute
$\phi_1(s)=\psi(s)+\lambda\int_a^b R(s,r;\lambda)\psi(r)dr$

Therefore,

$$\phi_1(s)=\sin(\pi s)+\frac{3\pi A-\pi-3}{3\pi}s+\frac{3}{2}s\int_0^1 r(\sin(\pi r)+\frac{3\pi A-\pi-3}{3\pi}r)dr. \tag{33}$$

Simplifying the resulting integral gives:

$$\phi_1(s)=\sin(\pi s)+\frac{9\pi A-3\pi}{6}\pi s. \tag{34}$$

Now to find the value of$\phi_2(s)$ we substitute the value A, and $\phi_1(s)$ which are defined in Eq.(34) and Eq.(29) in Eq.(28), reads

$$\phi_2(s)=\frac{-2}{\pi}s+3As^2+s\int_0^1(\sin(\pi r)+\frac{9\pi A-3\pi}{6\pi}r)dr, \tag{35}$$

and

$$\phi_2(s)=3As^2+\left[\frac{9\pi A-3\pi}{12\pi}\right]s. \tag{36}$$

Substituting Eq.(36) in Eq.(29) we obtain

$A=\int_0^1(3Ar^2+\frac{9\pi A-3\pi}{12\pi}r)dr.$

Then $A=\frac{1}{3}$
And substituting$(A=\frac{1}{3})$ into equations (34) and (36), we get the exact solution

$\phi_1(s)=\sin(\pi s)$, and $\phi_2(s)=s^2$.

Example 4.2: Consider the coupled systems of SFIDEs

$\phi_1''(s) = -\frac{4}{3} + \int_0^1 [r\phi_1(r) + 3\phi_2(r)]dr$
$\phi_2''(s) = -\frac{2}{3}s + \frac{5}{3} + \int_0^1 [r\phi_1(r) + 2s\phi_2(r)]dr$
$\phi_1'(0) = 1, \quad \phi_1(0) = 0)$
$\phi_2'(0) = 0, \quad \phi_2(0) = 0.$

By using the Resolvent kernel method on Example 4.2 and integrating Eq.(36) from 0 to s, we get

$$\begin{aligned} \phi_1(s) &= -\frac{2}{3}s^2 + s + \frac{1}{2}s^2\int_0^1 [r\phi_1(r) + 3\phi_2(r)]dr \\ \phi_2(s) &= -\frac{1}{9}s^3 + \frac{5}{6}s^2 + \int_0^1 [\frac{1}{2}s^2 r\phi_1(r) + \frac{1}{3}s^3\phi_2(r)]dr \end{aligned} \tag{37}$$

$$\begin{aligned} A &= \int_0^1 \phi_2(r)dr \\ B &= \int_0^1 r\phi_1(r)dr. \end{aligned} \tag{38}$$

Inserting the value of A, defined by Eq.(38), into $\phi_1(s)$as given in Eq.(37), gives:

$$\phi_1(s) = \left(\frac{3A}{2} - \frac{2}{3}\right)s^2 + s + \frac{1}{2}\int_0^1 s^2 r\phi_1(r)dr. \tag{39}$$

Rewriting Eq.(39) in the standard form

$$\phi_1(s) = \psi(s) + \lambda\int_a^b k(s,r)\phi_1(r)dr.$$

Where

$\psi(s) = (\frac{3A}{2} - \frac{2}{3})s^2 + s, \lambda = \frac{1}{2}, k(s,r) = s^2 r \quad and a = 0, b = 1.$
The iterated kernels $k_n(s,r)$ is defined by:

$$k_n(s,r) = \begin{cases} s^2 r & n = 1 \\ \int_0^1 k(s,z)k_{n-1}(z,r)dz = (\frac{1}{4})^{n-1}s^2 r & n \geq 2 \end{cases} \tag{40}$$

and

$$R(s,r;\lambda) = \frac{8}{7}s^2 r. \tag{41}$$

To find the solution of $\phi_1(s)$, we insert the $R(s,r;\lambda) = \frac{8}{7}s^2 r$ into the following formula

$$\phi_1(s) = \psi(s) + \lambda\int_a^b R(s,r;\lambda)\psi(r)dr.$$

This implies that

$$\phi_1(s) = \left(\frac{3A}{2} - \frac{2}{3}\right)s^2 + s + \frac{4}{7}s^2\int_0^1 \left(\left(\frac{3A}{2} - \frac{2}{3}\right)r^3 + r^2\right)dr, \tag{42}$$

and

$$\phi_1(s) = \left(\frac{72A - 24}{6}\right)s^2 + s. \tag{43}$$

Now, inserting the value of B, defined by Eq.(41), into $\phi_2(s)$ as given in Eq.(40), gives

$$\phi_2(s) = -\frac{1}{9}s^3 + \left(\frac{B}{2} + \frac{5}{6}\right)s^2 + \frac{1}{3}\int_0^1 s^3\phi_2(r)dr. \tag{44}$$

Rewriting Eq.(44) in the standard form

$$\phi_2(s) = \psi(s) + \lambda\int_a^b k(s,r)\phi_2(r)dr, \tag{45}$$

where

$\psi(s) = -\frac{1}{9}s^3 + (\frac{B}{2} + \frac{5}{6})s^2, \lambda = 1/3, k(s,r) = s^3$
and $a = 0, b = 1.$
The iterated kernels $k_n(s,r)$ are given by

$$k_n(s,r) = \begin{cases} s^3 & n = 1 \\ \int_0^1 k(s,z)k_{n-1}(z,r)dz = (\frac{1}{4})^{n-1}s^3 & n \geq 2 \end{cases} \tag{46}$$

and

$$R(s,r;\lambda) = \sum_{n=1}^{\infty}\lambda^{n-1}k_n(s,r) = \sum_{(}n = 1)^{\infty}(\frac{1}{3})^{n-1}(\frac{1}{4})^{n-1}s^3 = \frac{12}{11}s^3. \tag{47}$$

Inserting Eq.(47) into the following, expression $\phi_1(s)$ to find

$\phi_2(s) = \psi(s) + \lambda\int_a^b R(s,r;\lambda)\psi(r)dr.$
We obtain

$$\phi_2(s) = -\frac{1}{9}s^3 + \left(\frac{B}{2} + \frac{5}{6}\right)s^2 + \frac{4}{11}s^3\int_0^1 (-\frac{1}{9}r^3 + \left(\frac{B}{2} + \frac{5}{6}\right)r^2)dr, \tag{48}$$

and

$$\phi_2(s) = -\frac{1}{9}s^3 + \left(\frac{B}{2} + \frac{5}{6}\right)s^2 + \frac{4B + 6}{66}s^3. \tag{49}$$

To compute A and B, we insert the formulas for $\phi_1(s)$ and $\phi_2(s)$ from Eqs.(43) and (49) into the system given by Eq.(36), which gives

$$\begin{aligned} A &= 1/3, \\ B &= 1/3. \end{aligned} \tag{50}$$

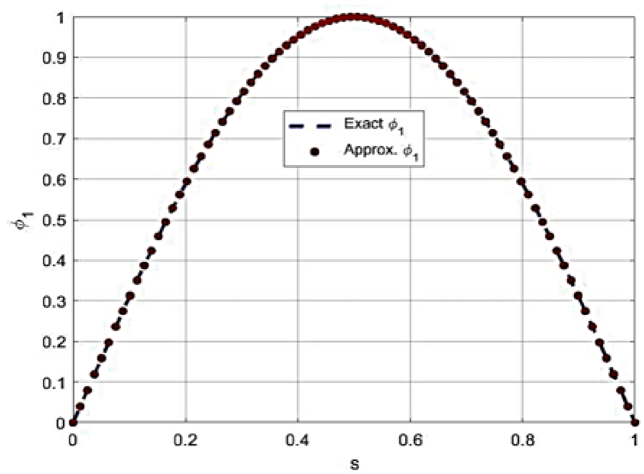

Figure 1. Numerical solution for Example 4.1, for $\phi_1(s)$

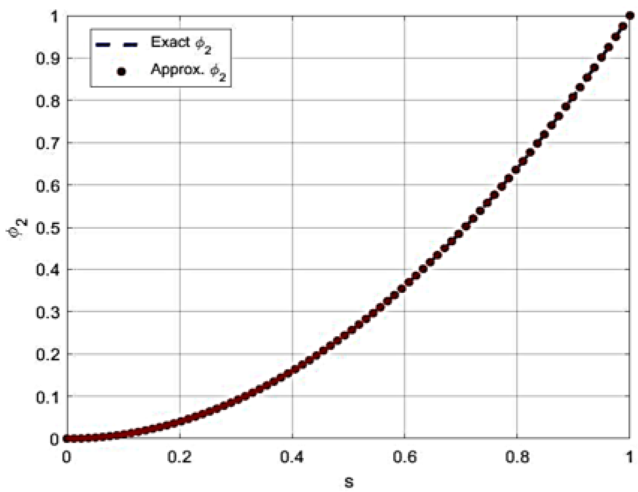

Figure 2. Numerical solution for Example 4.1,for $\phi_2(s)$

To obtain the exact solution, we substitute the values of A and B, given into Eq.(50), into Eqs.(43) and (49), and we obtain

$\phi_1(s) = s$, and $\phi_2(s) = s^2$.

4.2 Numerical Solutions:

In this section, we present a series of numerical experiments to assess the effectiveness of the proposed numerical method. The computed results are compared against the analytical solutions introduced in Section 2. Specifically, $\phi_1^{Exactsol.}$, and $\phi_2^{Exact.sol}$ denote the exact solutions, while $\phi_1^{Appro.sol}$, $\phi_2^{Appro.sol.}$ represent the corresponding approximate solutions for$\phi_1(s)$ and $\phi_2(s)$, respectively. To evaluate the accuracy of the method, the absolute error is computed as: $Absolute \quad Error = |\phi_i^{Exact \quad sol.} - \phi_i^{Approx. \quad sol.}|, i = 1,2$

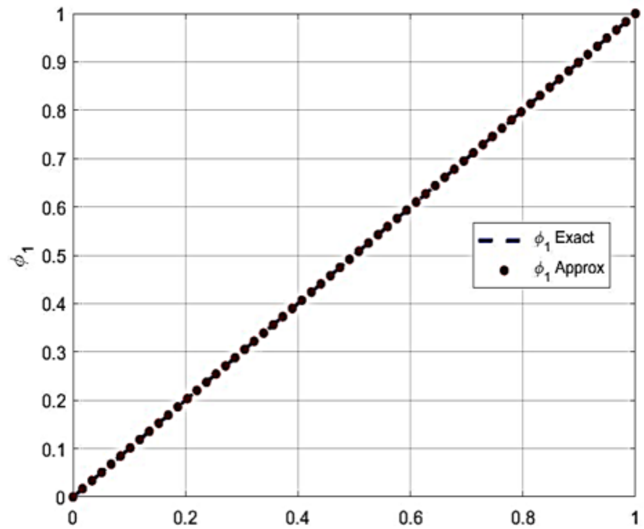

Figure 3. Numerical solution for Example 4.2, for $\phi_1(s)$

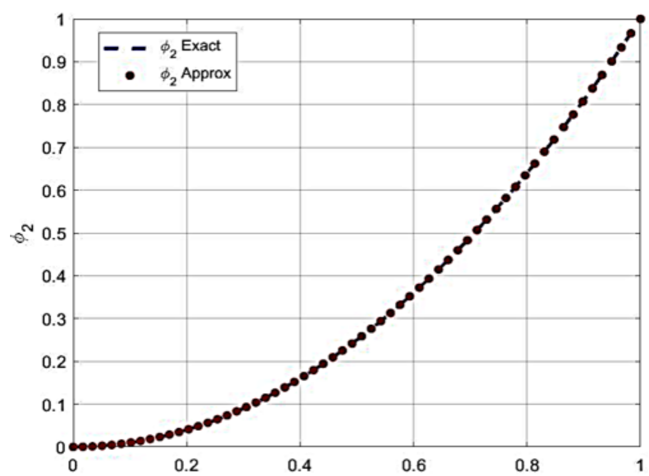

Figure 4. Numerical solution for Example 4.2,for $\phi_2(s)$

Next, we explore the Haar wavelet collocation method, detailed in Section 2, by demonstrating it with Example 4.2.

The absolute error statistics in Tables 1 and 2 clearly demonstrate that the numerical solutions obtained using the Haar-suggested colocation method matches the relevant analytical solutions in every evaluation case. These smaller error amplitudes supported the high level of precision achieved by numerical model. Additionally, the characteristics of the solution are shown in Figures 1, 2, 3 and 4, together with the uniform mail age distributions related to it. These illustrations provide an additional view of the system's physical behaviour by capturing precise dynamics of the solution across selected spatial points.

In addition to roughly following the trend of analytical solutions, approximate solutions also show stability and con-

Table 1. Comparison of Approximate and Exact Solutions for Example 5.1 Using HWM and RKM.

s_i	$\phi_1^{Appro.\quad sol.}$	$\phi_1^{Exact\quad sol.}$	Absolute Error	$\phi_2^{Appro.\quad sol.}$	$\phi_2^{Exact\quad sol.}$	Absolute Error
7.81e-03	2.45e-02	2.45e-02	2.36e-05	4.43e-05	6.10e-05	1.67e-05
2.34e-02	7.34e-02	7.35e-02	7.08e-05	3.79e-04	5.49e-04	1.70e-04
3.90e-02	1.22e-01	1.22e-01	1.18e-04	1.20e-03	1.52e-03	3.20e-04
5.46e-02	1.70e-01	1.70e-01	1.65e-04	2.52e-03	2.99e-03	4.65e-04
7.03e-02	2.18e-01	2.19e-01	2.12e-04	4.33e-03	4.94e-03	6.07e-04
8.59e-02	2.66e-01	2.66e-01	2.59e-04	6.63e-03	7.38e-03	7.45e-04
1.01e-01	3.13e-01	3.13e-01	3.06e-04	9.43e-03	1.03e-02	8.80e-04
1.17e-01	3.59e-01	3.59e-01	3.52e-04	1.27e-02	1.37e-02	1.01e-03
1.32e-01	4.04e-01	4.05e-01	3.99e-04	1.65e-02	1.76e-02	1.13e-03
1.48e-01	4.49e-01	4.49e-01	4.45e-04	2.07e-02	2.20e-02	1.25e-03

Table 2. Comparison of Approximate and Exact Solutions for ϕ_1 and ϕ_2 in Example 4.2 Using HWCM and RKM.

s_i	$\phi_1^{Appro.\quad sol.}$	$\phi_1^{Exact\quad sol.}$	Absolute Error	$\phi_2^{Appro.\quad sol.}$	$\phi_2^{Exact\quad sol.}$	Absolute Error
7.81e-03	7.81e-03	7.81e-03	3.22e-07	6.11e05	6.10e-05	8.19e-08
2.34e-02	2.34e-02	2.34e-02	2.90e-06	5.50e-04	5.49e-04	7.40e-07
3.90e-02	3.90e-02	3.90e-02	8.07e-06	1.52e-03	1.52e-03	2.07e-06
5.46e-02	5.47e-02	5.46e-02	1.58e-05	2.99e-03	2.99e-03	4.10e-06
7.03e-02	7.03e-02	7.03e-02	2.61e-05	4.95e-03	4.94e-03	6.85e-06
8.59e-02	8.59e-02	8.59e-02	3.90e-05	7.39e-03	7.38e-03	1.03e-05
1.01e-01	1.01e-01	1.01e-01	5.45e-05	1.03e-02	1.03e-02	1.45e-05
1.17e-01	1.17e-01	1.17e-01	7.26e-05	1.37e-02	1.37e-02	1.95e-05
1.32e-01	1.32e-01	1.32e-01	9.33e-05	1.76e-02	1.76e-02	2.54e-05
1.48e-01	1.48e-01	1.48e-01	1.16e-04	2.20e-02	2.20e-02	3.20e-05

sistency of various spatial locations, highlighting the effectiveness of the technique in handling the complexity of the integro-differential equation of Fredholm combined equations. Visual comparisons reinforce the numerical results by highlighting slight deviations from the exact curves, even in regions where the solution exhibts rapid fluctuations. In conclusion, the strong correlation between numerical and precise solutions, supported by a quantitative error analysis and a visual inspection, analytic demonstrates the accuracy, robustness, and reliability of the Haar collocation technique. These results support the proposed technique as a strong and effective tool for solving the Fredholm systems with combined integro-difference, particularly in cases where the analytical solutions prove difficult or impossible to obt.

5. Conclusions :

The main goal of this paper is to solve a system of integro-differential Fredholm equations analytically and numerically. We use the Haar wavelet collocation approach to numerically solve those equations, which is an efficient discretization scheme for dealing with such problems. The trapezoidal approach is used to approximate the integral term because it is simple and efficient in numerical integration. Theoretical error estimates are obtained to evaluate the correctness of the numerical solution, offering information on the effectiveness and convergence behaviour of the suggested approach.

The correctness and efficiency of the numerical results are also assessed by a series of calculations compared to the corresponding analytical solutions. According to the results, the Haar wavelet collocation approach is reliable and accurate, accurately capturing the key features of the Fredholm integro-differential equations and producing higher-precision numerical solutions with great accuracy.

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Data Availability Statement: All of the data supporting the findings of the presented study are available from corresponding author on request.

Declarations:

Conflict of interest: The authors declare that they have no conflict of interest.

Ethical approval: Since there were no human or animal participants in this study, ethical approval was not required in compliance with national and institutional rules and regulations.

Author contributions:

Ibrahim Qadr Salim, Younis A. Sabawi, and Mohammad Sh. Hasso contributed to the conceptualization, formal analysis, and preparation of the original draft, as well as to the review and editing of the manuscript.

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تقنيات النواة المحـلة ومويجات هار لحل المعادلات التكامليةالتفاضلية المترابطة من نوع فريدهولم

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الخلاصة

تركز هذه الدراسة على معادلات فريدولم التكامليةالتفاضلية، والتي تُستخدم بشكل متكرر في مجالات الرياضيات التطبيقية، والفيزياء، والهندسة. ولمعالجة هذه الأنظمة والحصول على حلول دقيقة في ظل شروط مناسبة، نقترح طريقةً تحليليًا جديدًا. اخترنا طريقة التمركز باستخدام مويجات هار (*Haar wavelet collocation method*) نظرًا لبساطتها وفعاليتها وقدرتها على التعامل مع الحلول غير الملساء. أما الحدود التكاملية في هذه المعادلات، فقد تم حسابها باستخدام قاعدة شبه المنحرف، التي توفر توازنًا فعالًا بين الدقة والكفاءة الحسابية. وقد تم مقارنة النتائج مع الحلول التحليلية، وأظهرت المقارنات أن الاستراتيجية المقترحة توفر نتائج دقيقة للغاية وتشكل إطارًا قويًا لحل معادلات فريدولم التكامليةالتفاضلية.

الكلمات الدالة : نظام معادلات فريد هولم التكامليةالتفاضلية، موجات هار، نواة المحلّ

التمويل: لايوجد.

بيان توفر البيانات: جميع البيانات الداعمة لنتائج الدراسة المقدمة يمكن طلبها من المؤلف المسؤول.

اقرارات:

تضارب المصالح: يقر المؤلفون أنه ليس لديهم تضارب في المصالح.

الموافقة الأخلاقية: لم يتضمن هذا البحث أي تجارب على البشر أو على الحيوانات، بالتالي لم يكن من الضروري الحصول على موافقة أخلاقية.

مساهمات المؤلفين: ساهم إبراهيم قادر سليم، ويونس عبد سبعاوي، ومحمد شامي حسو في صياغة المفاهيم والتحليل الشكلي وإعداد المسودة الأصلية، بالإضافة إلى مراجعة وتحرير المخطوطة.