Existence Solutions for a Singular Nonlinear Problem with Dirichlet Boundary Conditions on Exterior Domains

Mageed Ali 1*, Joseph Iaia 2

1* Department of Mathematics, College of Science, University of Kirkuk, Kirkuk, Iraq.
2 Department of Mathematics, University of North Texas, Denton, Texas, USA.

*Corresponding author: mageedali@uokirkuk.edu.iq

Abstract

This paper has proved the existence of solutions that solve the Nonlinear Partial differential equation. A study of dynamical systems has developed on the exterior of the ball centered at the origin in \( \mathbb{R}^N \) with radius \( R > 0 \), with Dirichlet boundary conditions \( u = 0 \) on the boundary, and \( u(x) \) approaches 0 as \(|x|\) approaches infinity, where \( f(u) \) is local Lipschitzian singular at zero, and grows superlinearly as \( u \) approaches infinity, by introducing various scalings to elucidate the singular behavior near the center and at infinity. Also, \( N > 2 \), \( f(u) \sim |u|^{-q-1}u \) for small \( u \) with \( 0 < q < 1 \), and \( f(u) \sim |u|^{p-1}u \) for large \(|u| \) with \( p > 1 \). In addition, \( K(x) \sim |x|^{-\alpha} \) with \( 2 < \alpha < 2(N-1) \) for large \(|x| \). The fixed point method and other techniques have been used to prove the existence.

1. Introduction:

Certainly, exploring solutions to partial differential equations is crucial in various scientific disciplines, especially in physical mathematics[1, 2]. The existence and uniqueness of solutions, particularly in second-order PDEs with specified initial conditions, form a fundamental aspect of this field [3, 4, 5]. The existence of a positive solution of (1) on \( \mathbb{R}^N \) with \( K(r) \equiv 1 \) has been studied extensively [6, 7, 8, 9, 10, 11].

Recently, the exterior domain \( \mathbb{R}^N \setminus B_R(0) \) has been studied in [12, 13, 14, 15, 16, 17]. Since we are interested in the topic, it comes from the recent papers [16, 18, 11] that have been studied to find the solutions to differential equation problems on exterior domains.

In [19], was studied (1)–(3) with \( K(r) r^{-\alpha} \), where \( f \) is singular at 0 and grows sublinearly at infinity, with different values of \( \alpha \). Also, in [20], the singular semilinear problem has infinitely many solutions on exterior domain. This article has proved the existence of solutions when \( f \) is singular at 0 and grows superlinearly at infinity.

This paper deals with the problem:

\[
\Delta u + K(|x|) f(u) = 0, \quad x \in \mathbb{R}^N \setminus B_R
\]

(1)

\[
u = 0 \text{ on } \partial (\mathbb{R}^N \setminus B_R)
\]

(2)

\[
u \to 0 \text{ as } |x| \to \infty
\]

(3)

where \( \Delta \) is the Laplacian operator, \( u : \mathbb{R}^N \to \mathbb{R} \), with \( N > 2 \), \( B_R \) is the ball of radius \( R > 0 \) centered at the origin in \( \mathbb{R}^N \) and \( K(x) > 0 \).
In addition, we suppose:

\[ f \text{ is an odd function, increasing on } (0, \infty), \]
\[ f \text{ is locally Lipschitz, } \exists \beta > 0 \text{ such that } f < 0 \text{ on } (0, \beta), \]
\[ f > 0 \text{ on } (\beta, \infty). \]

We assume:

\[ f(u) = -\frac{1}{|u|^{q-1}u} + g_1(u) \]

where \(0 < q < 1\) for small \(u\) and \(g_1(0) = 0\)
and:

\[ f(u) = |u|^{p-1}u + g_2(u) \]

where \(p > 1\) for large \(u\) and \(\lim_{u \to +\infty} \frac{g_2(u)}{|u|^p} = 0.\)

Also, we assume \( F(u) = \int_0^u f(s) \, ds. \) We know that \(f\) is odd it implies that \(F\) is even and from (H2) it follows that \(f\) is integrable near \(u = 0\). Thus \(F\) is continuous and \(F(0) = 0\). It also follows that \(F\) is bounded below and from (H1), \(\exists \gamma\) with \(0 < \beta < \gamma\) such that:

\[ F < 0 \text{ on } (0, \gamma), F > 0 \text{ on } (\gamma, \infty), \text{ and } F > F_0 \text{ on } \mathbb{R}. \]

We also suppose \(K\) and \(K'\) are continuous function on \([R, \infty)\) with:

\[ K(r) > 0, \exists \alpha \in (2, 2(N-1)) \text{ such that } \lim_{r \to \infty} \frac{r K'}{K} = -\alpha, \]
and so \(2(N-1) + \frac{r K'}{K} > 0.\)

In addition, we assume \(\exists K_1 > 0, K_2 > 0\) such that:

\[ \frac{K_1}{r^\alpha} \leq K(r) \leq \frac{K_2}{r^\alpha} \text{ on } [R, \infty). \]

**2. Preliminaries:**

We are interested to study existence solutions of (1)–(3), we rewrite the equation with \(r = |x|\), \(u(r) = u(|x|)\) where \(u\) satisfies:

\[ u''(r) + \frac{N - 1}{r} u'(r) + K(r)f(u(r)) = 0 \quad \text{on } (R, \infty), \]

\[ u(R) = 0, \quad u'(R) = a > 0. \]

To emphasize the dependence on the initial parameter \(a\), we denote the solution by \(u_a(r)\). Since \(f(u)\) is not continuous at \(u = 0\), here we can not apply the usual existence-uniqueness theorem for ordinary differential equations and so we have to prove the existence of a solution of equations (4)–(5) on \([R, R + \varepsilon]\) for some \(\varepsilon > 0\) by using a different method.

First rewrite equation (4) as

\[ (r^{N-1} u_a'(r))' + r^{N-1} K(r)f(u_a(r)) = 0, \]
then integrate over \([R, r]\) and use \(u_a'(R) = a\).

This gives:

\[ r^{N-1} u_a'(r) - a R^{N-1} + \int_R^r r^{N-1} K(r)f(u_a(s)) \, ds \, dr = 0. \]

Multiply above by \(r^{-(N-1)}\), integrate again over \([R, r]\) and use \(u(R) = 0\) gives:

\[ u_a(r) = a R^{N-1} \left[ \frac{r^{2-N} - R^{2-N}}{2-N} \right] - \int_R^r \frac{1}{r^{N-1}} \int_r^s s^{N-1} K(s)f(u_a(s)) \, ds \, dt \quad \text{for } r \in (R, \infty). \]

Now let \(w(r) = u_a(r) / r^R\) so \(w(r) = (r - R)w(r)\) and

\[ w(R) = \lim_{r \to R^+} \frac{u_a(r)}{r^R} = u_a'(R) = a. \]

Rewriting (6) we get:

\[ w(r) = a R^{N-1} \left[ \frac{r^{2-N} - R^{2-N}}{2-N} \right] - \frac{1}{r - R} \int_R^r \frac{1}{r^{N-1}} \int_r^s s^{N-1} K(s) f((s-R)w(s)) \, ds \, dt. \]

We use the fixed point method to solve (7). Let define:

\[ A = \left\{ w \in C[R, R+\varepsilon] \text{ with } w(R) = a > 0 \text{ and } \right\} 
\[ |w(r) - a| \leq \frac{a}{2} \text{ on } [R, R+\varepsilon]. \]
where $C[R, R + \varepsilon]$ is continuous functions on $[R, R + \varepsilon]$ with $\varepsilon > 0$.

Let:

$$
||w|| = \sup_{x \in [R, R + \varepsilon]} |w(x)|.
$$

Therefore $(A, ||.||)$ is a Banach space.

Now we define a map $T$ on $A$ by $Tw(r) = a$ and:

$$
Tw(r) = \frac{aR^{N-1}}{2 - N} \left[ \frac{r^2 - N - R^2 - N}{r - R} \right] - \frac{1}{r - R} \int_{R}^{r} \int_{R}^{s} s^{N-1} K(s) f((s - R)w(s)) \, ds \, dt \quad \text{for} \quad r > R.
$$

We will prove that $T$ is a principle contraction mapping with $T(w) \in A$ for each $w \in A$ if $\varepsilon > 0$ is sufficiently small. By using L'Hôpital's rule it follows that

$$
\lim_{r \to R^+} \frac{aR^{N-1}}{2 - N} \left[ \frac{r^2 - N - R^2 - N}{r - R} \right] = a.
$$

In addition, by (H2), by L'Hôpital's rule and $0 < q < 1$ we have:

$$
\lim_{r \to R^+} \frac{\int_{R}^{r} \int_{R}^{s} s^{N-1} K(s) f((s - R)w(s)) \, ds \, dt}{r - R} = 0.
$$

Therefore $\lim_{r \to R^+} Tw(r) = a$, and it follows that:

$$
|Tw(r) - a| \leq \frac{a}{2} \text{ on } [R, R + \varepsilon] \text{ if } \varepsilon > 0 \text{ is sufficiently small.}
$$

Thus We next show that $T$ is a contraction from $A$ into itself for sufficiently small $\varepsilon$.

For any $w_1, w_2 \in A$ we have:

$$
Tw_1(r) - Tw_2(r) = -\frac{1}{r - R} \int_{R}^{r} \int_{R}^{s} s^{N-1} K(s) \left[ f((s - R)w_1(s)) - f((s - R)w_2(s)) \right] \, ds \, dt.
$$

For $u \geq 0$ and by (H2) we know that $f(u) = -u^{-q} + g_1(u)$ so $f((s - R)w(s)) = -(s - R)^{-q}w^{-q}(s) + g_1((s - R)w(s))$ where $0 < q < 1$.

Then we first estimate:

$$
|f((s - R)w_1(s)) - f((s - R)w_2(s))| = \left| -\frac{1}{(s - R)^q} \left[ \frac{1}{w_1^q} - \frac{1}{w_2^q} \right] \right| + g_1((s - R)w_1(s)) - g_1((s - R)w_2(s))
$$

$$
\leq \frac{1}{(s - R)^q} \left[ \frac{1}{w_1^q} - \frac{1}{w_2^q} \right] + L|s - R||w_1 - w_2| \quad \text{where } L \text{ is the Lipschitz constant for } g_1 \text{ near } u = 0.
$$

Applying the mean value theorem to the right-hand side of (9) we get:

$$
\left[ \frac{q}{(s-R)^q} \right] + L|s - R||w_1 - w_2| = 2
$$

Using (10) in (8) and assuming $r \in [R, R + \varepsilon)$ gives:

$$
|Tw_1 - Tw_2| \leq \frac{1}{r - R} \int_{R}^{r} \int_{R}^{s} s^{N-1} K(s) |w_1 - w_2| \left[ \frac{q}{(s-R)^q} \right] + L \varepsilon \, ds \, dt
$$

$$
\leq K(R) \frac{|w_1 - w_2|}{r - R} \int_{R}^{r} \int_{R}^{s} s^{N-1} K(s) \left[ \frac{q}{(s-R)^q} \right] + L \varepsilon \, ds \, dt
$$

$$
\leq K(R) \frac{|w_1 - w_2|}{r - R} \left[ \frac{q(\frac{q}{a})^{q+1} + \varepsilon}{(2 - q)(1 - q)} + \varepsilon^2 \right].
$$

Since:

$$
\lim_{\varepsilon \to 0} \frac{q(\frac{q}{a})^{q+1} + \varepsilon}{(2 - q)(1 - q)} + \varepsilon^2 \frac{L}{2} = 0
$$

and $c = K(R) \frac{q(\frac{q}{a})^{q+1} + \varepsilon}{(2 - q)(1 - q)} + \varepsilon^2 \frac{L}{2}$, we can choose small enough $\varepsilon > 0$ satisfies that $0 < c < 1$ such that $T$ is a contraction on $C[R, R + \varepsilon]$.

So there exists a unique solution $w \in A$ with $Tw = w$ on $[R, R + \varepsilon]$ for some $\varepsilon > 0$.

Thus $u_\varepsilon(r) = (r-R)w(r)$ is a solution of (4)--(5) on $[R, R + \varepsilon]$ for some $\varepsilon > 0$.

Now let:

$$
E_\varepsilon(r) = \frac{u_\varepsilon^2(r)}{2K(r)} + F(u_\varepsilon).
$$

Using (4) and (H5) we get:

$$
E_\varepsilon'(r) = -\frac{u_\varepsilon^2(r)}{2K(r)} + \frac{rK'}{K} \leq 0.
$$
It follows that $E$ is non-increasing so:

$$E_a(r) = \frac{1}{2} u_a^2(r) + F(u_a) \leq \frac{1}{2} a^2 \leq E_a(R) \quad \text{for } r \geq R.$$  \hfill (13)

Since $F$ is bounded from below by (H4), so from (13) it implies that $u_a'$ and $u_a$ are uniformly bounded on $[R, \infty)$ and so existence follows wherever they are defined. We know that $f(u)$ is undefined at $u = 0$, so the solution of (4)–(5) exists as long as $u_a(r) > 0$. In addition, if $u_a(r_0) = 0$ but $u_a'(r_0) \neq 0$, we can use the same argument as on the previous page to establish existence of a solution of (4)–(5) in a neighborhood of $r_0$. If there is an $r_0$ such that $u_a(r_0) = 0$ and $u_a'(r_0) = 0$ then we show in the appendix that we can extend this solution to a neighborhood of $r_0$. Continuing this process we can find the existence of a solution of (4)–(5) on $[R, \infty)$.

**Lemma 2.1:** Let $u_a(r)$ solves (4)–(5) and assume that $2 < \alpha < 2(N-1)$. If $a$ sufficiently small, then $u_a(r) > 0 \ \forall r \in (R, \infty)$.

**Proof:** From (5) we have $u_a(R) = 0$ and $u_a'(R) = a > 0$. If $u_a'(r) > 0 \ \forall r \in (R, \infty)$ then $u_a(r) > 0 \ \forall r \in (R, \infty)$. So we are done in this case.

If $u_a(r)$ is not always greater than zero on $(R, \infty)$, then $u_a$ has a zero at $z_a$, and $u_a(r) > 0$ on $(R, z_a)$. In addition, exist $M_a$ such that $R < M_a < z_a$, where $M_a$ is a local maximum of $u_a$ with $u_a(M_a) > 0$ and $u_a'(r) > 0$ on $(R, M_a)$. From (4) we then have $u_a'(M_a) = 0$, $u_a''(M_a) \leq 0$ so $f(u_a(M_a)) \geq 0$ so $u_a(M_a) \geq \beta > 0$.

We now show $\lim_{a \to 0^+} M_a = +\infty$. Assume by the way of contradiction $\lim_{a \to 0^+} M_a \neq +\infty$. Then $\exists M^* > 0$ and a subsequence (still labeled $M_a$) such that $\lim_{a \to 0^+} M_a = M^*$.

Since $R \leq M_a \leq z_a$ then $0 \leq E_a(z_a) \leq E_a(M_a) \leq E_a(R)$.

Thus $0 \leq F(u_a(M_a)) \leq \frac{1}{2} \frac{\alpha^2}{K(r)}$ and so $\lim_{a \to 0^+} F(u_a(M_a)) = 0$. Since we know from earlier $u_a(M_a) \geq \beta > 0$ it follows then that:

$$\lim_{a \to 0^+} u_a(M_a) = \gamma. \quad \text{(14)}$$

On the interval $[R, z_a]$ it follows from (13) that:

$$0 \leq E_a(z_a) \leq E_a(r) = \frac{1}{2} u_a^2(r) + F(u_a(r)) \leq \frac{1}{2} \frac{\alpha^2}{K(r)} \to 0$$

as $a \to 0^+$ on $[R, z_a]$.

and as we saw earlier $u_a, u_a'$ are uniformly bounded on $[R, M^* + 1]$. Thus there exists a subsequence still labeled $u_a$ such that $u_a$ is uniformly convergent on $[R, M^* + 1]$ with

$$\lim_{a \to 0^+} u_a(r) = u^*(r) \text{ on } [R, M^* + 1] \text{ and } \lim_{a \to 0^+} u_a(M_a) = u^*(M^*) \text{ on } [R, M^* + 1].$$

Then from (14) we get $u^*(M^*) = \gamma$. Also since $u_a$ is increasing on $[R, M_a]$ it follows that $u^*$ is increasing on $[R, M^*]$ and:

$$0 \leq u^* \leq \gamma \text{ on } [R, M^*]. \quad \text{(16)}$$

Now consider the following identity which follows directly from (4):

$$r^{2(N-1)} \left[ \frac{1}{2} u_a^2(r) + K(r)F(u_a) \right]' = \left( r^{2(N-1)} K(r) \right)' F(u_a). \quad \text{(17)}$$

Integrating on $[R, r)$ gives:

$$r^{2(N-1)} \left[ \frac{1}{2} u_a^2(r) + K(r)F(u_a) \right] = \int_R^r \left( r^{2(N-1)} K(t) \right)' F(u_a) \ dt. \quad \text{(18)}$$

Since $a \to 0$ and $u_a \to u^*$ uniformly on $[R, M^* + 1]$ then taking the limit in (18) gives:

$$\lim_{a \to 0^+} \int_R^r \left( r^{2(N-1)} K(t) \right)' F(u_a) \ dt = \lim_{a \to 0^+} \int_R^r \left( r^{2(N-1)} K(t) \right)' F(u^*) \ dt. \quad \text{(19)}$$

Dividing by $r^{2(N-1)} K(r)$ gives:

$$\lim_{a \to 0^+} \frac{1}{2} \frac{u_a^2(r)}{K(r)} + F(u_a) = \lim_{a \to 0^+} \frac{\int_R^r \left( r^{2(N-1)} K(t) \right)' F(u^*) \ dt}{r^{2(N-1)} K(r)}. \quad \text{(19)}$$

Thus $\lim_{a \to 0^+} u_a^2$ exists and since $u_a' \geq 0$ on $[R, M_a]$ then $\lim_{a \to 0^+} u_a''$ exists and so $\lim_{a \to 0^+} u_a' = u^*.$

Combining this with (15) it follows that $\frac{1}{2} \frac{u_a^2(r)}{K(r)} + F(u_a(r)) \equiv 0$ on $[R, M^*]$ and then by (17) and (H5), $\left( r^{2(N-1)} K(t) \right)' F(u^*) \equiv 0$. Thus $F(u^*) \equiv 0$. Therefore $u^* = \text{constant}$ but since $u^*(M^*) = \gamma$ and $u^*(R) = 0 < \gamma$, we get a contradiction. Thus $M_a$ cannot be bounded and therefore:

$$\lim_{a \to 0^+} M_a = \infty. \quad \text{(20)}$$

Next for $M_a < r < z_a$ we have $0 \leq E_a(z_a) \leq E_a(r) \leq E_a(M_a) = F(u_a(M_a))$ thus $u_a(M_a) \geq \gamma$ and so:

$$\frac{1}{2} \frac{u_a^2(r)}{K(r)} + F(u_a(r)) \leq E(M_a) = F(u_a(M_a)) \quad r \geq M_a. \quad \text{(21)}$$
Rewriting and integrating (21) from $M_a$ to $z_a$, and changing variable gives:

$$
\int_0^\gamma \frac{dt}{\sqrt{2} \sqrt{F(u_a(M_a)) - F(r)}} \leq \int_0^{z_a(M_a)} \frac{dt}{\sqrt{2} \sqrt{F(u_a(M_a)) - F(r)}} \leq \frac{z_a}{M_a} \sqrt{K(r)} dr.
$$

(22)

Now using (H5)–(H6) and that $\alpha > 2$ gives:

$$
\int_{M_a}^{z_a} \sqrt{K(r)} dr \leq \int_{M_a}^{z_a} \sqrt{K_2 r - a} dr = \sqrt{K_2} \left( \frac{1 - \alpha^2}{2} - M_a^{1 - \frac{\alpha}{2}} \right)
$$

$$
\leq \frac{2 \sqrt{K_2}}{\alpha - 2} M_a^{1 - \frac{\alpha}{2}}.
$$

(23)

Thus combining (22) and (23) we obtain:

$$
\int_0^\gamma \frac{dt}{\sqrt{2} \sqrt{F(u_a(M_a)) - F(r)}} \leq \frac{2 \sqrt{K_2}}{\alpha - 2} M_a^{1 - \frac{\alpha}{2}}.
$$

(24)

Now taking the limit as $a \to 0^+$ in inequality (24) using (14), (20), and $\alpha > 2$ gives:

$$
0 < \int_0^\gamma \frac{dt}{\sqrt{2} \sqrt{F(r)}} \leq \lim_{a \to 0^+} \frac{2 \sqrt{K_2}}{\alpha - 2} M_a^{1 - \frac{\alpha}{2}} = 0.
$$

This is a contradiction. Thus $u_a(r) > 0$ on $[R, \infty)$ if $a > 0$ is sufficiently small. This completes the proof of Lemma 2.1.

Next we show that $u_a(r)$ has many zeros on $(R, \infty)$ as $a \to \infty$.

**Lemma 2.2:** Let $u_a(r)$ be the solution of (4)–(5) and suppose (H1)–(H6). Then $u_a(r)$ has a local maximum $M_a$ if $a$ is sufficiently large, $u_a(M_a) \to \infty$ as $a \to \infty$, and $M_a \to R^+$ as $a \to \infty$.

**Proof:** First, suppose $M_a$ is a positive local maximum. Then $u_a'(M_a) = 0$, $u_a''(M_a) \leq 0$ and from equation (4), we see $f(u_a(M_a)) \geq 0$ (since $K(M_a) > 0$) so $u_a(M_a) \geq \beta$. Thus $u_a$ cannot have a local maximum before $M_a$ reaches $\beta$.

Next, suppose by the way of contradiction that $0 \leq u_a \leq \beta$ for sufficiently large $a$ and all $r \in [R, \infty)$. Then we see $f(u_a(M_a)) \leq 0$ (since $K(M_a) > 0$) so $u_a(M_a) \geq \beta$. Hence $(r^{N-1} u_a')' \geq 0$ on $[R, r]$. Integrating on $[R, r]$ gives:

$$
\frac{r^{N-1} u_a'}{r^N} = a R^{N-1} - \frac{1}{2} u_a^2(r) \geq 0.
$$

(25)

Hence $u_a$ is increasing on $[R, r]$. Rewriting (25) and integrating gives:

$$
u_a(r) \geq a R^{N-1} \left[ \frac{r^{2-N} - R^{2-N}}{2-N} \right] = \frac{a R}{N-2} \left[ 1 - \left( \frac{r}{R} \right)^{N-2} \right]
$$

on $[R, r]$.

(26)

Then from (26) we see $u_a(2R) \geq \frac{a R}{N-2} \left[ 1 - \frac{1}{2^{2-N}} \right]$ and

$$
limit_{a \to \infty} \frac{a R}{N-2} \left[ 1 - \frac{1}{2^{2-N}} \right] = \infty
$$

which contradicts the assumption that $0 \leq U_a \leq \beta$. Thus if $a$ is sufficiently large then $u_a(r)$ gets larger than $\beta$.

Next we show max $u_a(r) \to \infty$ as $a \to \infty$. Suppose by way of contradiction that max $u_a(r) \leq B$ where $B$ does not depend on $a$ for $a$ large.

Since $r^{2(N-1)} K(r) F(u_a)$ and $(r^{2(N-1)} K(r))' F(u_a)$ are continuous on $[R, 2R]$ then $r^{2(N-1)} K(r) F(u_a) \leq A_1$ with $A_1 > 0$ and $\int_{R}^{2R} (r^{2(N-1)} K(r))' F(U_a) \leq A_2$ with $A_2 > 0$ so rewriting (18) we obtain:

$$
r^{2(N-1)} \frac{1}{2} u_a'^2(r) \geq \frac{R^{2(N-1)} a^2}{2} - [A_1 + A_2].
$$

(27)

Since the right-hand side of (27) goes to $\infty$ as $a \to \infty$ then we see there is a $C_a$ with $C_a > 0$ such that $\lim_{a \to \infty} C_a = \infty$ and:

$$
|u_a'| \geq \sqrt{\frac{2 C_a}{R^{2-N}}} > 0
$$

on $[R, 2R]$.

(28)

thus $u_a' > 0$ for $a$ sufficiently large $[R, 2R]$ and integrating (28) over $(R, 2R)$ we get:

$$
B \geq u_a(2R) \geq \sqrt{2 C_a} \left[ \frac{1 - \frac{2^2-N}{N-2}}{R^{2-N}} \right] R^{2-N}
$$

but $\lim_{a \to \infty} \sqrt{2 C_a} \left[ \frac{1 - \frac{2^2-N}{N-2}}{R^{2-N}} \right] R^{2-N} = \infty$ which is a contradiction to the fact that $u_a$ was bounded by $B$ on $[R, 2R]$. Thus

$$
\max_{[R, 2R]} u_a \to \infty
$$

as $a \to \infty$.

(29)

Now let us show that $u_a(r)$ has a local maximum $M_a$ if $a$ is sufficiently large. Suppose by the way of contradiction that $u_a$ is increasing for all $r > R$. Since it follows from (13) that $u_a$ is bounded then we see $\lim_{r \to \infty} u_a(r) = L_a$ with $L_a > 0$. Also since $E_a$ is non-increasing it follows that $\lim_{r \to \infty} \frac{1}{2} \frac{u_a^2}{R^{2-N}} + F(u_a(r))$ exists. Since $F(u_a) \to F(L_a)$ as $r \to \infty$ it then follows that.
\[ \lim_{r \to \infty} \frac{u^2}{2K(r)} \text{ exists. Dividing (18) by } r^{2(N-1)}K(r) \text{ we have:} \]
\[ \frac{1}{2} \frac{u^2}{K(r)} + F(u_a(r)) = \frac{R^{2(N-1)}a^2}{2r^{2(N-1)}K(r)} + \int_{r}^{\infty} \left( \frac{u'}{r^{2(N-1)}K(r)} \right)^2 F(u_a) \, dr. \]
\( (30) \)

By (H5)–(H6) it follows that \( r^{2(N-1)}K(r) \to 0 \) as \( r \to \infty \).

Then taking limits as \( r \) goes to infinity and using L'Hopital's rule in (30) we get:
\[ \lim_{r \to \infty} \frac{1}{2} \frac{u^2}{K(r)} + F(L_a) = 0 + F(L_a). \]
\( (31) \)

And so \( \lim_{r \to \infty} \frac{1}{2} \frac{u^2}{K(r)} = 0. \)

Next by assumption \( u_a(r) \) is increasing and \( s0 L_a \geq \max_{[K,2K]} u_a(r). \)

It follows then from (29) that
\[ \lim_{a \to \infty} L_a = \infty. \]
\( (32) \)

Since \( E_a \) is non-increasing and \( \frac{1}{2} \frac{u^2}{K(r)} \to 0 \) as \( r \to \infty \) then we see:
\[ \frac{1}{2} \frac{u^2}{K(r)} + F(u_a(r)) \geq F(u_a(L_a)), r \geq R. \]
\( (33) \)

Rewriting and integrating (33) over \([R, \infty)\) we get:
\[ \int_{0}^{L_a} \frac{dt}{\sqrt[2]{\sqrt{2}F(L_a)}} = \int_{R}^{\infty} \frac{|u'_a(r)|dr}{\sqrt{2F(L_a) - F(t)}} \geq \int_{R}^{\infty} \sqrt[2]{K(r)} \, dr \]
\( (34) \)

From right-hand side of (34) since \( \alpha > 2 \) and using (H6) we get:
\[ \int_{R}^{\infty} \sqrt[2]{K(r)} \geq \int_{R}^{\infty} K1 \frac{a^2}{2} = \frac{2K1}{\alpha - 2} R^{1 - \frac{\alpha}{2}}. \]
\( (35) \)

Thus we get:
\[ \int_{0}^{L_a} \frac{dt}{\sqrt[2]{\sqrt{2}F(L_a) - F(t)}} \geq \frac{2K1}{\alpha - 2} R^{1 - \frac{\alpha}{2}}. \]
\( (36) \)

Finally let us show that \( \lim_{a \to \infty} \int_{0}^{L_a} \frac{dt}{\sqrt[2]{\sqrt{2}F(L_a) - F(t)}} = 0 \) which contradicts and thus our assumption that \( u_a \) is increasing is false and therefore \( u_a \) must have a local maximum.
\[ \int_{0}^{L_a} \frac{dt}{\sqrt[2]{\sqrt{2}F(L_a) - F(t)}} = \int_{0}^{L_a} \frac{dt}{\sqrt[2]{\sqrt{2}F(L_a) - F(t)}} + \int_{0}^{L_a} \frac{dt}{\sqrt[2]{\sqrt{2}F(L_a) - F(t)}}. \]
\( (37) \)

From (32) we know \( L_a \to \infty \) as \( a \to \infty \) and so it follows from (H3) that \( \lim_{a \to \infty} \frac{f}{L_a} = 0 \) thus for a large \( \frac{L_a}{2} \) is large then \( F(t) < F\left(\frac{L_a}{2}\right) \) also \( F(L_a) - F\left(\frac{L_a}{2}\right) \). \( (38) \)

By the mean value theorem there is \( \alpha \) such that \( \frac{L_a}{2} < \frac{d_1}{L_a} \) then \( F(L_a) - F\left(\frac{L_a}{2}\right) = f(d_1) \frac{L_a}{2} \). \( (39) \)

Since \( f \) is increasing for \( a \) large then \( f\left(\frac{L_a}{2}\right) \leq f(d_1) \) so
\[ \lim_{a \to \infty} \frac{L_a}{2} \geq \frac{\sqrt{2L_a}}{\sqrt{2F(L_a) - F\left(\frac{L_a}{2}\right)}}. \]
\( (40) \)

Thus by (38), (39), and (40) then:
\[ \lim_{a \to \infty} \int_{0}^{L_a} \frac{dt}{\sqrt[2]{\sqrt{2}F(L_a) - F(t)}} = 0. \]
\( (41) \)

Second, we estimate \( t \in \left[ \frac{L_a}{2}, L_a \right] \) we have \( F \) is continuous and \( f \) is increasing so by the mean value theorem there is a \( d_2 > 0 \) with \( \frac{L_a}{2} < d_2 < L_a \) so \( F(L_a) - F(t) = f(d_2) |L_a - t| \geq f\left(\frac{L_a}{2}\right) |L_a - t| \) rewrite the second part of (37) we get:
\[ \int_{\frac{L_a}{2}}^{L_a} \frac{dt}{\sqrt[2]{\sqrt{2}F(L_a) - F(t)}} \leq \int_{\frac{L_a}{2}}^{L_a} \frac{dt}{\sqrt[2]{\sqrt{2}f\left(\frac{L_a}{2}\right)(L_a - t)}} \]
\( (42) \)

Taking limit as \( a \) goes to infinity and by (H3) and (35)
\[ \lim_{a \to \infty} \frac{L_a}{2} \geq \frac{\sqrt{2L_a}}{\sqrt{2f\left(\frac{L_a}{2}\right)}}. \]
\( (43) \)

Thus (42) and (43) gives:
\[ \lim_{a \to \infty} \int_{\frac{L_a}{2}}^{L_a} \frac{dt}{\sqrt[2]{\sqrt{2}F(L_a) - F(t)}} = 0. \]
\( (44) \)

Combining (41) and (44) with (37) we have:
\[ \int_{0}^{L_a} \frac{dt}{\sqrt[2]{\sqrt{2}F(L_a) - F(t)}} = 0. \]
\( (45) \)
Now taking limits in (36) we get: $K \frac{r_1^{\frac{q}{2}} - r_2^{\frac{q}{2}}}{r_1 - r_2} \leq 0$ which is false. Thus $u_a$ must have a first local maximum $M_a$ if $a$ is sufficiently large.

Next we show that $u_a(M_a) \geq \max u_a$. Since $u_a$ has a first local maximum $M_a$. Case 1: if $M_a > 2R$. Since $u_a$ is increasing on $[R, M_a]$ then $u_a(M_a) \geq u_a(2R) = \max u_a$ so we do this case. case 2: if $R < M_a < 2R$. Suppose by way of contradiction there is $t_0$ with $M_a < t_0 < 2R$ such that $u_a(t_0) > u_a(M_a)$ then there is a smallest $s_0$ with $s_0 > M_a$ such that $u_a(s_0) = u_a(M_a)$ then for $M_a < r < s_0$ we have $F(u_a(M_a)) = E(s_0) \leq E(r) \leq E(M_a) = F(u_a(M_a))$ since $\frac{1}{2} \int_0^{u_a(s_0)} K(r) = 0$ and $F(u_a(M_a)) = F(u_a(s_0))$ therefore $E(r)$ is a constant on $[M_a, s_0]$ thus $E'(r) = 0$ then $u_a'(r) \equiv 0$ on $[M_a, s_0]$. By the uniqueness of the solution of the initial value problem we have $u_a'(r) \equiv 0$ on $[R, \infty)$ but we know $u'(R) = a > 0$ which is a Contradiction. So no $t_0$ exists. Thus $u_a(M_a) \geq \max u_a$ and $\max u_a \to \infty$ as $a \to \infty$. Thus $\lim u_a(M_a) = \infty$.

Now let us show $\lim M_a = R$. Since $E_a(r)$ is non-increasing it follows that if $R \leq r \leq M_a$ then:

$$\frac{1}{2} \int_0^{M_a} K(r) + F(u_a(r)) \geq F(u_a(M_a)) \text{ on } [R, M_a].$$

Rewriting, integrating over $(R, M_a)$ and changing variables we get:

$$\int_0^{M_a} \frac{u_a'(r)}{\sqrt{2 \sqrt{F(u_a(M_a))} - F(r) - 1}} dr = \int_R^{M_a} \frac{u_a'(r)}{\sqrt{2 \sqrt{F(u_a(M_a))} - F(u_a(r))}} \geq \int_R^{M_a} \sqrt{K(r)} dr.$$

From the right-hand side of (46) using (H6) we get:

$$\int_R^{M_a} \sqrt{K(r)} \geq \int_R^{M_a} \sqrt{K_1 r^{-\alpha}} = \sqrt{K_1} \left( M_a^{\frac{\alpha}{2} - 1} - R^{\frac{\alpha}{2} - 1} \right)$$

(47)

since $\alpha > 2$. It follows from (45) that the left-hand side of (46) goes to 0 as $a \to \infty$ therefore it follows from (47) that $M_a \to R$ as $a \to \infty$.

This completes the proof of lemma.

**Lemma 2.3:** Suppose (4)–(5) and $N \geq 2$. Let $u_a(r)$ be the solution of (4)–(5) and suppose $2 < \alpha < 2(N - 1)$. Then $u_a(r)$ has at least $n$ zeroes on $(0, \infty)$ if $a$ sufficiently large.

**Proof:** Let $V(r) = u_a(r + M_a)$, then $V(0) = u_a(M_a), V'(r) = u_a'(r + M_a)$ and $V''(r) = u_a''(r + M_a)$. Substituting in equation (4) we get:

$$u_a''(r + M_a) + \frac{N - 1}{r + M_a} u_a'(r + M_a) + K(r + M_a) f(u_a(r + M_a)) = 0$$

so $V''(r) + N \frac{1}{r + M_a} V'(r) + K(r + M_a) f(V(r)) = 0$ with $V(0) = u_a(M_a)$ and $V'(0) = 0$. Now if we replace $r$ with $\frac{r}{\lambda}$ where $\lambda > 0$ then we get:

$$V''\left( \frac{r}{\lambda} \right) + N \frac{1}{\lambda + M_a} V'(\frac{r}{\lambda}) + K\left( \frac{r}{\lambda} + M_a \right) f\left( V\left( \frac{r}{\lambda} \right) \right) = 0.$$ 

(48)

Now let:

$$W_2(r) = \lambda r^{\frac{\alpha}{2}} V\left( \frac{r}{\lambda} \right) = \lambda r^{\frac{\alpha}{2}} u_a\left( \frac{r}{\lambda} + M_a \right).$$

Then:

$$W_2'(r) = \lambda r^{\frac{\alpha}{2} - 1} V'\left( \frac{r}{\lambda} \right) = \lambda r^{\frac{\alpha}{2} - 1} u_a'\left( \frac{r}{\lambda} + M_a \right)$$

$$W_2''(r) = \lambda r^{\frac{\alpha}{2} - 2} V''\left( \frac{r}{\lambda} \right) = \lambda r^{\frac{\alpha}{2} - 2} u_a''\left( \frac{r}{\lambda} + M_a \right)$$

and substituting above in (49) we get:

$$W_2''(r) + \frac{N - 1}{\lambda + M_a} W_2'(r) + \lambda r^{\frac{\alpha}{2}} K\left( \frac{r}{\lambda} + M_a \right) \left[ W_2 |^{p+1} W_2 + \frac{2}{2(p+1)} \right] = 0.$$ 

(50)

simplifying (51) we get:

$$W_2''(r) + \frac{N - 1}{\lambda + M_a} W_2'(r) + \lambda r^{\frac{\alpha}{2}} K\left( \frac{r}{\lambda} + M_a \right) \left[ W_2 |^{p+1} W_2 + \frac{2}{2(p+1)} \right] = 0.$$

(52)

where $G(u) = \int_0^u g_2(r) dr$. Then:

$$E_\lambda(r) = \frac{W_2^2}{2 \lambda (\frac{r}{\lambda} + M_a)} K\left( \frac{r}{\lambda} + M_a \right) K'\left( \frac{r}{\lambda} + M_a \right) \leq 0.$$ 

(53)

where the bracketed term is greater than or equal to 0 by (H5). It follows from (53) that $E_\lambda$ is non-increasing and so:

$$E_\lambda(r) = \frac{1}{2} \lambda \frac{W_2^2}{K\left( \frac{r}{\lambda} + M_a \right) K\left( \frac{r}{\lambda} + M_a \right)} + \frac{W_2 |^{p+1} W_2 + G\left( \lambda r^{\frac{\alpha}{2}} W_2(r) \right)}{2(p+1)} \leq \frac{1}{P+1} + \frac{G\left( \lambda r^{\frac{\alpha}{2}} W_2(r) \right)}{2(p+1)} = E_\lambda(0).$$

Using (H3):
\[
\lim_{\lambda \to \infty} G(\lambda, r^2) = 0
\]
so for \( \lambda \) sufficiently large we get:
\[
\frac{1}{2} \frac{W_\lambda^2}{K(\frac{r}{\lambda} + M_a)} + \frac{|W_\lambda|^{p+1}}{P+1} + \frac{G(\lambda, r^2 W_\lambda(r))}{\lambda^{\frac{2(p+1)}{p+1}}} \leq \frac{1}{P+1} + \frac{1}{P+1}
\]
\[
= \frac{2}{P+1}
\]
so:
\[
\frac{1}{2} \frac{W_\lambda^2}{K(\frac{r}{\lambda} + M_a)} + \frac{|W_\lambda|^{p+1}}{P+1} \leq \frac{2}{P+1} - \frac{G(\lambda, r^2 W_\lambda(r))}{\lambda^{\frac{2(p+1)}{p+1}}}.
\] (57)

Using (H3) it follows that \( \lim_{u \to \infty} \frac{|G(u)|}{|u|^{p+1}} = 0. \)

So:
\[
\frac{G(u)}{|u|^{p+1}} \leq \frac{1}{2(P+1)} \frac{|u|^{p+1}}{P+1} .
\]

Also since \( G(u) \) is continuous when \( |u| \leq C_0 \) then there is \( D \) so that \( |G(u)| \leq D \) when \( |u| \leq C_0 \) and so
\[
|G(u)| \leq D + \frac{1}{2(P+1)} |u|^{p+1} \forall u \in \mathbb{R}.
\] (58)

Thus:
\[
\left| G\left( \lambda, r^2 W_\lambda(r) \right) \right| \leq D + \frac{1}{2(P+1)} \left| \lambda, r^2 W_\lambda(r) \right|^{p+1} = D + \frac{1}{2(P+1)} \lambda^{\frac{2(p+1)}{p+1}} |W_\lambda(r)|^{p+1}
\] (59)

Substituting (59) into (57) gives:
\[
\frac{1}{2} \frac{W_\lambda^2}{K(\frac{r}{\lambda} + M_a)} + \frac{|W_\lambda|^{p+1}}{P+1} \leq \frac{2}{P+1} + \frac{D}{\lambda^{\frac{2(p+1)}{p+1}}} + \frac{1}{2(P+1)}
\]
\[
|W_\lambda(r)|^{p+1}
\]
so:
\[
\frac{1}{2} \frac{W_\lambda^2}{K(\frac{r}{\lambda} + M_a)} + \frac{|W_\lambda|^{p+1}}{2(P+1)} \leq \frac{2}{P+1} + \frac{D}{\lambda^{\frac{2(p+1)}{p+1}}} \leq \frac{2}{P+1} + 1
\]

for \( \lambda \) sufficiently large. Thus \( W_\lambda \) and \( W'_\lambda \) are uniformly bounded on compact sets. So by Arzela-Ascoli, there is a subsequence still labeled \( \lambda \) such that \( W_\lambda \to W^* \) uniformly on compact sets and so \( W^* \) is continuous. It can be shown in a similar argument as (59) that:
\[
\lim_{u \to \infty} k\left( R + M_a \right) \lambda^{\frac{2p}{p+1}} g_2\left( \lambda, r^2 \right) = 0
\]

since \( \frac{g_2(u)}{u^p} \to 0 \) as \( u \to \infty \) so \( g_2(u) < \varepsilon \) if \( u \geq L \) then
\[
g_2(u) < \varepsilon |u|^p \text{ if } u \geq L \text{ thus } g_2(u) \leq D_1 + \varepsilon |u|^p
\]

so
\[
\left| k\left( R + M_a \right) \lambda^{\frac{2p}{p+1}} g_2\left( \lambda, r^2 \right) \right| \leq \left| k\left( R + M_a \right) \lambda^{\frac{2p}{p+1}} \right| D_1 + \varepsilon \lambda^{\frac{2p}{p+1}} + \varepsilon k\left( R + M_a \right)
\]

is also uniformly bounded. Then it follows from (51) that
\( W_\lambda \) is also uniformly bounded. Thus \( W'_\lambda \to W^* \) uniformly on compact sets. Then taking limits in (51) we get:
\[
(W^*)^p + K(R) |W^*|^{p+1} = 0
\] (60)

with \( W^*(0) = 1, W^*(0) = 0. \) Thus:
\[
\frac{1}{2}(W^*)^2 + K(R) |W^*|^{p+1} = \frac{K(R)}{P+1}.
\] (61)

It follows from (61) that \( W^* \leq 1. \) We now show \( W^* \) has an infinite number of zeros on \([0, \infty)\). Suppose \( (W^*)' \leq 0 \) for all \( r \geq R. \) Then \( W^* \) is bounded and decreasing so:
\[
\lim_{r \to \infty} W^*(r) = L.
\] (62)

Taking limits in (61) gives:
\[
\lim_{r \to \infty} \frac{1}{2} W^* r^2 (r) + K(R) \frac{|L|^{p+1}}{P+1} = \frac{K(R)}{P+1}.
\] (63)

so:
\[
\lim_{r \to \infty} \left| (W^*)' (r) \right| = \sqrt{\frac{2K(R)}{P+1} \left[ 1 - |L|^{p+1} \right]}.
\] (65)

Thus for large \( r \) and \( r_0 \)
\[
\int_{r_0}^{r} -(W^*)' (r) dr = \int_{r_0}^{r} \left| (W^*)' (r) \right| dr \geq \frac{1}{2} \int_{r_0}^{r} \sqrt{\frac{2K(R)}{P+1} \left[ 1 - |L|^{p+1} \right]} dr
\] (66)
we get:

\[
-W^*(r) + W^*(r_0) \geq \frac{1}{2} \sqrt{\frac{2K(R)}{P+1} \left[ 1 - |L|^{p+1} \right]} (r - r_0) \to \infty
\]

as \( r \to \infty \).

(67)

on the left-hand side of (67) is bounded which contradicts that \( W^* \) is bounded. Thus \( |L| = 1 \) and since \( W^*(0) = 1 \) and since \( W^{**} < 0 \) then \( W^* \) is decreasing near \( r = 0 \) also \( W^{*'}(0) = -K(R) < 0 \) so \( W^* \) is not constant so \( L \neq 1 \) and thus \( W^{*'} \leq 0 \) then \( L = -1 \).

\[
\frac{1}{2} (W^*)^2 + \frac{K(R)}{P+1} |W^*|^{p+1} = \frac{K(R)}{P+1}
\]

(68)

\[
(W^*)' = \sqrt{\frac{2K(R)}{P+1} \left[ 1 - (W^*)^{p+1} \right]}
\]

(69)

\[
\int_0^r \frac{-W^*(r) dr}{\sqrt{1 - |W^*(r)|^{p+1}}} = \int_0^r \frac{|W^*| (r) dr}{\sqrt{1 - W^*^{p+1}}} = \int_0^r \sqrt{\frac{2K(R)}{P+1}}
\]

\[
= \sqrt{\frac{2K(R)}{P+1} \left[ 1 - (W^*)^{p+1} \right]}
\]

(70)

if we make change of variable \( t = W^*(r) \) and \( dt = W^*(r) dr \) we get:

\[
\int_{W^*(r)}^1 \frac{dt}{\sqrt{1 - t^{p+1}}} = \int_{0}^r \frac{\sqrt{2K(R)} dr}{P+1} \to \infty \text{ as } r \to \infty
\]

(71)

if \( W^*(r) \geq 0 \) and \( (W^*)'(r) \leq 0 \) since \( |W^*(r)| \) bounded by 1, so \( W^*(r) \to -1 \) as \( r \to \infty \).

\[
\int_{-1}^{1} \frac{dt}{\sqrt{1 - t^{p+1}}} = \infty
\]

(72)

but left-hand side is finite. This is a contradiction. Thus \( W^* \) must have a first local minimum \( m^* \). Let \( r = m^* \) in (65) so \( |W^*|^{p+1}(m^*) = 1 \) so \( W^*(m^*) = \pm 1 \) but since \( W^*(0) = 1 \) and \( W^* \) is initially decreasing then it follows that \( W^*(m^*) = -1 \) so \( W^* \) has a first zero \( Z_1 \) and we can show \( W(m^* + t) = W(m^* - t) \) therefore \( W \) is periodic with period \( 2m^* \) so \( W \) has infinite many zeros. This completes the proof.

In this paper, we prove the following:

**Theorem 1:** Assuming (H1)–(H6) then there exists a solutions of (1)–(3).

### 3. The Main Results:

Let \( S_0 = \{ a > 0 \mid u_a(r) > 0 \ \forall \ r > R \} \). By Lemma 2.1 we know that if \( a > 0 \) and \( a \) is sufficiently small then \( u_0 > 0 \) for all \( r > R \). Thus \( S_0 \) is nonempty. By Lemma 2.3 we see that if \( a \) sufficiently large then \( u_0 \) has a zero. Hence \( S_0 \) is bounded from above. So the supremum of \( S_0 \) exists and let \( a_0 = \sup S_0 > 0 \).

**Lemma 3.1:** \( u_{a_0}(r) > 0 \) for \( r > R \) and \( \lim_{r \to \infty} u_{a_0}(r) = 0 \).

**Proof:** Suppose first by the way of contradiction that \( u_{a_0}(r) \) is not positive for \( r > R \). So there exists \( Z_0 > R \) such that \( u_{a_0}(Z_0) = 0 \) and \( u_{a_0}(r) > 0 \) on \((R, Z_0)\).

Assume \( u_{a_0}'(Z_0) < 0 \) So there is \( r_1 > Z_0 \) such that \( u_{a_0}(r_1) < 0 \). We also know that \( u_{a_0}'(r) \) varies continuously with \( a \). Thus on any compact set \( K_0 \), \( \lim_{a \to a_0} u_a(r) = u_{a_0}(r) \) uniformly on \( K_0 \).

So if \( a \) is close enough to \( a_0 \) then \( u_{a_0}(r_1) < 0 \).

In particular if \( 0 < a < a_0 \) then \( u_{a}(r_1) < 0 \), but this contradicts that then \( u_a(r) > 0 \) for \( r > R \) and \( 0 < a < a_0 \).

Therefore \( u_{a_0}(r) \) does not have a zero. So \( u_{a_0}(r) > 0 \) for \( r > R \).

For \( a > a_0 \) then \( u_a(r) \) has a zero \( z_a \). We now show \( \lim_{a \to a_0} Z_a = \infty \), because otherwise if there is a \( B > 0 \) such that \( z_a \leq B \) for all \( a \) close to \( a_0 \) then there is a subsequence still labeled \( a \) such that \( Z_a \to Z^* \).

Also since \( E_{a}(r) \leq \frac{a^2}{2K(R)} \leq \frac{1}{2} (a_0 + 1)^2 \) for all \( r \geq R \) then \( u_a \) and \( u'_a \) are uniformly bounded on \([R, a_0 + 1] \) and so for further subsequence still labeled \( a \) we have \( u_a \to u_{a_0} \) uniformly on compact sets so \( \lim_{a \to a_0} u_a(Z_a) = u_{a_0}(Z^*) \).

So \( u_{a_0}(Z^*) = 0 \) but we showed earlier \( u_{a_0}(r) > 0 \) for \( r > R \). This is a contradiction. Thus \( \lim_{a \to a_0} Z_a = +\infty \).

In addition, we now show \( E_{a_0}(r) \geq 0 \) for all \( r > R \). Let us integrate the identity over \((r_0, r)\), we get:

\[
\int_{r_0}^{r} \left[ \left( r^{2(N-1)} \left( \frac{1}{2} u_a^2(r) + K(r) F(u_a(r)) \right) \right)' \right] = \left( r^{2(N-1)} K(r) \right)' F(u_a(r))
\]

rewriting

\[
= r^{2(N-1)} \left( \frac{1}{2} u_a^2(r) + K(r) F(u_a(r)) \right)' = \left( r^{2(N-1)} K(r) \right)'
\]

we get

\[
E_{a_0}(r) = \int_{r_0}^{r} \left( \frac{1}{2} u_a^2(r) + K(r) F(u_a(r)) \right) dr = \left( r^{2(N-1)} K(r) \right)'
\]

Suppose by the way of contradiction suppose there is \( r_1 > R \) such that \( E_{a_0}(r_1) < 0 \). Again by continuous depen
dence of the $u_0(r)$ and $u'_0(r)$ on the parameter $a$ we get $E_0(r_1) < 0$ if $a$ is close enough to $a_0$. On the other hand, if $a > a_0$ then $u_0$ has a first zero $z_a$ and $U_a > 0$ for $R < r < z_a$ and since $E_{a_0}(r_1) < 0$ and $E_{a_0}$ is non-increasing then $z_a \leq r$, thus $0 \leq E_0(z_a) \leq E_0(r_1) < 0$ where $z_a < r_1$. But $z_a \to \infty$ as $a \to a_0$ therefore $E_{a_0}(r) \geq 0 \forall r \geq R$.

**Lemma 3.2:** $u_{a_0}(r)$ has a local maximum $M_{a_0} > R$.

**Proof:** Suppose not. Then $u'_{a_0}(r) \geq 0 \forall r > R$. Also \( \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)} + F(u_{a_0}(r)) \) and $E_{a_0}(r) \leq E_{a_0}(R) = \frac{d^2}{dK(R)}$. It follows from this that $u_{a_0}$ is bounded and since $u'_{a_0} \geq 0$ then $\lim_{r \to \infty} u_{a_0}(r) = L > 0$.

Since $E_{a_0}$ is non-increasing then for all $r > R$ then from (H4) it follows that \( F(u_{a_0}) \) is bounded from below and since \( \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)} \geq 0 \) then \( \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)} + F(u_{a_0}) \) is bounded from below and thus $\lim_{r \to \infty} \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)} + F(u_{a_0}(r))$ exists. Also since $u_{a_0} \to L$ it follows that $\lim_{r \to \infty} F(u_{a_0}(r)) = F(L)$ and so it follows that $\lim_{r \to \infty} \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)}$ exists. Now let us show \( \lim_{r \to \infty} \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)} = 0 \). Consider the following identity which follows from (4) and integrating over $(r, r_0)$ we get:

\[
\int_{r_0}^{r} \left( \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)} + K(r)F(u_{a_0}(r)) \right) dr = \int_{r_0}^{r} \left( \frac{r^2}{2} \right) F(u_{a_0}(r)) dr
\]

so

\[
\frac{1}{2} \frac{d^2 u^2(r_0)}{dK(r)} + F(u_{a_0}(r_0)) = \frac{C_0}{K(r_0)^{2(N-1)}} + \frac{1}{r_0} \int_{r_0}^{r} \left( \frac{r^2}{2} \right) K(r)F(u_{a_0}(r)) dr
\]

for some constant $C_0$. Taking the limit as $r$ goes to infinity and using (H6) then $\lim_{r \to \infty} \frac{C_0}{K(r)^{2(N-1)}} = 0$ so using L'Hopital rule

\[
\lim_{r \to \infty} \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)} + F(u_{a_0}(r)) = \lim_{r \to \infty} \frac{1}{r_0} \int_{r_0}^{r} \left( \frac{r^2}{2} \right) K(r)F(u_{a_0}(r)) dr
\]

so $\lim_{r \to \infty} \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)} = 0$. Also from lemma 3.1, $E_{a_0} \geq 0$ and since $0 \leq E_{a_0} \to F(L)$ it follows $L \geq \gamma$.

Next we return to (4) \( - \left( r^N - u'_{a_0}(r) \right)^2 + r^{N-1} K(r) F(u_{a_0}(r)) \) since $L \geq \gamma$ and $f(u_{a_0}) \geq 0$. Since $u_{a_0}$ is increasing and $u_{a_0}(r) \to L \geq \gamma$ as $r \to \infty$ then for large $u_{a_0}(r) \geq r^{2N-3} \beta > \beta$ then there exists $C_1 > 0$ such that $f(u_{a_0}) \geq C_1 > 0$ for $r$ sufficiently large we get:

\[
- \left( r^{N-1} u'_{a_0}(r) \right)^2 + C_1 r^{N-1} K(r) \geq 0 \]

Integrating over $(r_0, r)$ where $r_0, r$ are sufficiently large then we get:

\[
\int_{r_0}^{r} \left( \frac{r^{N-1}}{2} u'_{a_0}(r) \right)^2 + C_1 r^{N-1} K(r) \leq 0
\]

so

\[
r^{N-1} u'_{a_0}(r) - r_0^{N-1} u'_{a_0}(r_0) + C_1 \frac{r^{N-\alpha - r_0^{N-\alpha}}}{N-\alpha} \leq 0 \text{ if } 2 < \alpha
\]

$< N$ then \( r^{N-1} u'_{a_0}(r) - r_0^{N-1} u'_{a_0}(r_0) + C_1 \frac{r^{N-\alpha} - r_0^{N-\alpha}}{N-\alpha} \to -\infty
\]

Since $r_0^{N-1} u'_{a_0}(r_0) = \text{constant}$ and $\lim_{r \to \infty} r^{N-\alpha} = +\infty$, then $\lim_{r \to \infty} r^{N-1} u'_{a_0}(r) = -\infty$ so $u'_{a_0}$ must get negative. Thus $u_{a_0}$ has a local max $M_{a_0}$.

Now we show that $u'_{a_0}(r) \leq 0$ for $r > M_{a_0}$. If not then there is $r_1 > M_{a_0}$ such that $u'_{a_0}(r_1) > 0$ so $u_{a_0}$ has a local min $m_{a_0} > M_{a_0}$ such that $u'_{a_0}(m_{a_0}) = 0$ and $u'_{a_0}(m_{a_0}) \geq 0$.

so $f(u_{a_0}(m_{a_0})) \leq 0$, but $0 < u_{a_0}(m_{a_0}) \leq \beta$. From lemma 3.1 we have $0 \leq E_{a_0}(m_{a_0}) = F(u_{a_0}(m_{a_0}))$.

\[
u_{a_0}(m_{a_0}) \geq \gamma, \text{ but this is a contradiction. Since } 0 < u_{a_0}(m_{a_0}) \leq \beta < \gamma \text{. Thus } u'_{a_0} < 0 \text{ for all } r > M_{a_0}. \text{ Since } u_{a_0} > 0 \text{ then } \lim_{r \to \infty} u_{a_0}(r) = A \text{ with } A \geq 0 \text{ for } r > R.
\]

We will show that $A = 0$. We know $E_{a_0}(r)$ is non-increasing and bounded below so: $\lim_{r \to \infty} E_{a_0}(r)$ exists, and $\lim_{r \to \infty} \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)} = 0$ exists $0 \leq E_{a_0}(r) = \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)} + F(u_{a_0}(r))$. Taking limit of $E_{a_0}(r)$ as $r$ goes to infinity.

\[
\lim_{r \to \infty} E_{a_0}(r) = F(A)
\]

so $0 \leq F(A)$ so since $A \geq 0$ then either $A = 0$ or $A \geq \gamma$. Let us assume $A \geq \gamma$ by above we get: $0 \leq \lim_{r \to \infty} E_{a_0}(r) = \lim_{r \to \infty} \frac{1}{2} \frac{d^2 u^2(r)}{dK(r)} + F(A)$.

So $A = 0$ and thus $\lim_{r \to \infty} u_{a_0}(r) = 0$ so $u_{a_0}$ is solution of (4)–(5).

**4. Conclusions:**

Through this work, we have been able to prove the existence of a solution to the singular superlinear Dirichlet problem (1) on the exterior domain in $R^N$. When $f$ is singular
at zero and $f$ grows superlinear at infinity, the proof we presented here seems to have some techniques for localized solutions. Also, we show that the energy is strictly decreasing.

**A. Appendix**

**Lemma 1:** Let $z > 0$. There is a solution $U_a$ of equation (4) if $u_a(z) = u'_a(z) = 0$ on $(z, z + \varepsilon)$ for some $\varepsilon > 0$.

**Proof:** Suppose first that $u_a(0) = 0$ and $u'_a(z) = 0$ with $u_a \in C^2(R, z - \varepsilon, \varepsilon)$ and $u_a \in C^0(R, z - \varepsilon)$. Let us determine the behavior of $u_a(r)$ on $(z - \varepsilon, z)$.

Using the fact that $f(u_a) = \frac{-1}{u_a^{1+q}} + g_1(u_a)$ where $0 < q < 1$, $g_1(0) = 0$ and $g_1$ is continuous at $u_a = 0$ then multiplying (4) by $|u_a|^{q-1} u_a$ we obtain:

$$|u_a|^{q-1} u_a u''_a(r) + \frac{N-1}{r} |u_a|^{q-1} u_a u'_a(r) + K(r)(-1 + g_1(u_a)) u_a^{q-1} u_a = 0.$$  

(73)

Since $g_1$ is continuous at $u_a = 0$ with $0 < q < 1$ then

$$\lim_{r \to z^-} K(r) g_1(u_a) |u_a|^{q-1} u_a = 0.$$  

Also since $u'_a$ is continuous with $u'_a(z) = 0$ and $0 < q < 1$ then $\lim_{r \to z^-} \frac{1}{2} |u_a|^{q-1} u_a u'_a = 0$ therefore from (73) this implies

$$\lim_{r \to z^-} |u_a|^{q-1} u_a u''_a(r) = K(z).$$

In addition, since $\lim_{r \to z^-} \frac{1}{2} u_a^2 = 0$ and $\lim_{r \to z^-} \frac{1}{1-q} |u_a|^{1-q} = 0$ then by L’Hospital’s rule we have:

$$K(z) = \lim_{r \to z^-} |u_a|^{q-1} u_a u''_a(r)$$

$$= \lim_{r \to z^-} \left( \frac{1}{1-q} |u_a|^{1-q} \right)^2$$

$$= \lim_{r \to z^-} \frac{1}{1-q} |u_a|^{1-q}.$$  

Thus $\lim_{r \to z^-} |u_a|^{q-1} u_a u''_a(r)$ is close to $z$ gives:

$$\int_{r}^{z} \left( \sqrt[2-q]{K(z) - \delta} \right) ds < \int_{r}^{z} \frac{u_a^{q-1}}{u_a^{1+a}} \left( \frac{2}{1-q} K(z) + \delta \right) ds$$

$$\left( \frac{2}{1-q} K(z) - \delta \right) (z - r) < \frac{2}{q+1} u_a^{q+1} < \left( \frac{2}{1-q} K(z) + \delta \right) (z - r)$$

so

$$\left( \frac{2}{1-q} K(z) - \delta \right) \leq \frac{2}{q+1} u_a^{q+1} \leq \left( \frac{2}{1-q} K(z) + \delta \right)$$

on $(z, z + \varepsilon)$.

Thus:

$$\lim_{r \to z^-} \frac{u_a^{q+1}}{(z - r)^{\frac{q+1}{q}}} = \frac{q + 1}{2} \frac{2}{1-q} K(z).$$

Let

$$W(r) = \frac{u_a(r)}{(z - r)^{\frac{q+1}{q}}}$$

so

$$\lim_{r \to z^-} W(r) = \left[ \frac{q + 1}{2} \frac{2}{1-q} K(z) \right]^{\frac{2}{q+1}}$$

so we define

$$W(z) = \lim_{r \to z^-} W(r) = \lim_{r \to z^-} \frac{u_a(r)}{(z - r)^{\frac{q+1}{q}}} = \left[ \frac{q + 1}{2} \frac{2}{1-q} K(z) \right]^{\frac{2}{q+1}}.$$  

This tells us how $u_a$ behaves on $(z - \varepsilon, z)$ so we expect $U$ to behave similarly on $(z, z + \varepsilon)$ so we will try now to prove the existence of a solution on $(z, z + \varepsilon)$ so that:

$$\lim_{r \to z^-} \frac{u_a}{(z - r)^{\frac{q+1}{q}}} = \left[ \frac{q + 1}{2} \frac{2}{1-q} K(z) \right]^{\frac{2}{q+1}}.$$  

(74)

Now assuming such a solution exists, Rewriting (4), integrating over $(z, r)$, and using $u'_a(z) = 0$ we get:

$$r^{N-1} u'_a(r) = - \int_{z}^{r} \frac{1}{r^{N-1}} \int_{z}^{s} s^{N-1} K(s) f(u_z) ds dt$$

(75)

Multiplying (75) by $\frac{1}{r^{N-1}}$, integrating over $(z, r)$ and using (H1) gives:

$$u_a(r) = - \int_{z}^{r} \frac{1}{r^{N-1}} \int_{z}^{s} s^{N-1} K(s) \left[ \frac{-1}{u_a^{q-1}} u_a(s) + g_1(u_a) \right] ds dt.$$  

(76)

Making the change of variables of (76):

$$u_a(r) = -(r - z)^{\frac{2}{q+1}} W(r).$$
Then (76) becomes:

\[
(r-z)^{q-1} W(r) = -\int_0^r \frac{1}{t^{q-1}} \int_z^t s^{N-1} K(s) \left[ \frac{-1}{(s-z)^{\frac{2q}{q-1}} W^{q-1} W} \right] ds dt \\
+ g_1 \left( -(s-z)^{\frac{2}{q-1}} W(s) \right) ds dt
\]

So:

\[
W(r) = \frac{-1}{(r-z)^{\frac{2q}{q-1}}} \int_0^r \frac{1}{t^{q-1}} \int_z^t s^{N-1} K(s) \left[ \frac{-1}{(s-z)^{\frac{2q}{q-1}} W^{q-1} W} \right] ds dt \\
+ g_1 \left( -(s-z)^{\frac{2}{q-1}} W \right) ds dt.
\]

Assuming \(W(r)\) is continuous at \(z\) then taking limits in (77) and using L'Hopital’s rule we get:

\[
W(z) = \lim_{r \to z^+} \int_0^r \frac{1}{t^{q-1}} \int_z^t s^{N-1} K(s) \left[ \frac{-1}{(s-z)^{\frac{2q}{q-1}} W^{q-1} W} \right] ds dt \\
+ g_1 \left( -(s-z)^{\frac{2}{q-1}} W \right) ds dt.
\]

Using L'Hopital’s rule again we get:

\[
W(z) = \frac{1}{q-1} \lim_{r \to z^+} \int_0^r \frac{1}{t^{q-1}} \int_z^t s^{N-1} K(r) \left[ \frac{-1}{(r-z)^{\frac{2q}{q-1}} W^{q-1} W} \right] ds dt \\
+ g_1 \left( -(s-z)^{\frac{2}{q-1}} W \right) ds dt
\]

simplifying above we get:

\[
W(z) = \frac{(q+1)^2 K(z)}{2(1-q)|W(z)|^{q-1} W(z)}
\]

and thus:

\[
|W(z)| = \left[ \frac{(q+1)^2 K(z)}{2(1-q)} \right]^{\frac{1}{q-1}}
\]

Let \(W(r) = CY(r)\) where \(C = \left( \frac{(q+1)^2 K(z)}{2(1-q)} \right)^{\frac{1}{q-1}}\) Then \(Y(z) = 1\) so

\[
Y(r) = \frac{-1}{C(r-z)^{\frac{2q}{q-1}}} \int_0^r \frac{1}{t^{q-1}} \int_z^t s^{N-1} K(s) \left[ \frac{-1}{(s-z)^{\frac{2q}{q-1}} C Y^q} \right] ds dt \\
+ g_1 \left( (s-z)^{\frac{2}{q-1}} C Y \right) ds dt.
\]

Now we attempt to can solve (78) by using the contraction mapping principle theorem. We define the set:

\[B = \{ Y \in C[z,z+\epsilon] \mid Y(z) = 1 \text{ and } ||Y(r) - 1|| < \delta \}\]

where \(\delta\) is sufficiently small.

Let:

\[||Y|| = \sup_{x \in [z,z+\epsilon]} |Y(x)|\]

Now define \(T : B \to C[z,z+\epsilon]\) by:

\[
TY(r) = \frac{-1}{C(r-z)^{\frac{2q}{q-1}}} \int_0^r \frac{1}{t^{q-1}} \int_z^t s^{N-1} K(s) \left[ \frac{-1}{(s-z)^{\frac{2q}{q-1}} C Y^q} \right] ds dt \\
+ g_1 \left( (s-z)^{\frac{2}{q-1}} CY \right) ds dt.
\]

Let us suppose \(Y_1, Y_2 \in B\) then:

\[
TY_1(r) - TY_2(r) = \frac{-1}{C(r-z)^{\frac{2q}{q-1}}} \int_0^r \frac{1}{t^{q-1}} \int_z^t s^{N-1} K(s) \left[ \frac{-1}{(s-z)^{\frac{2q}{q-1}} C Y^q} \right] ds dt \\
+ g_1 \left( (s-z)^{\frac{2}{q-1}} CY_1 \right) - g_1 \left( (s-z)^{\frac{2}{q-1}} CY_2 \right) ds dt.
\]

divided the integration in two parts:

For the first part of the integral since \(Y_1, Y_2 \in B\) .

Then by the mean value theorem there is \(Y_3\) between \(Y_1, Y_2\) also since \(0 < Y_2 < Y_3 < Y_1 < \delta + 1\) where \(|Y_i - 1| < \delta\) for \(i = 1, 2, 3\) then \(1 - \delta < Y_3 < 1 + \delta\) then \(\left[ \frac{1}{Y_i^q} - \frac{1}{Y_2^q} \right] = \frac{q}{Y_i^{q+1}} \frac{|Y_1 - Y_2|}{(1-\delta)^{q+1}} \leq \frac{q}{Y_i^{q+1}} |Y_1 - Y_2| \leq \frac{q}{Y_i^{q+1}} |Y_1 - Y_2|\).

Then the first part of the integral becomes:

\[
\left| \frac{-1}{C(r-z)^{\frac{2q}{q-1}}} \int_0^r \frac{1}{t^{q-1}} \int_z^t s^{N-1} K(s) \left[ \frac{-1}{(s-z)^{\frac{2q}{q-1}} Y^q} \right] ds dt \right|
\]

\[
\leq \frac{q}{(1+\delta)^{q+1} \max_{[z,z+\epsilon]} K(r)} |Y_1 - Y_2| \int_0^r \frac{1}{t^{q-1}} \int_z^t \frac{1}{(s-z)^{\frac{2q}{q-1}}} ds dt.
\]

Carrying out the integration and recalling \(C_Y^{q+1} = \frac{(q+1)^2 K(z)}{2(1-q)}\) we obtain:

\[
\leq \frac{q}{(1+\delta)^{q+1} \max_{[z,z+\epsilon]} K(r)} |Y_1 - Y_2| \int_0^r \frac{1}{t^{q-1}} \int_z^t \frac{1}{(s-z)^{\frac{2q}{q-1}}} ds dt.
\]

\[
\leq \frac{q}{(1+\delta)^{q+1} \max_{[z,z+\epsilon]} K(r)} |Y_1 - Y_2| \int_0^r \frac{1}{t^{q-1}} \int_z^t \frac{1}{(s-z)^{\frac{2q}{q-1}}} ds dt.
\]

\[
\leq \frac{q}{(1+\delta)^{q+1} \max_{[z,z+\epsilon]} K(r)} |Y_1 - Y_2| \int_0^r \frac{1}{t^{q-1}} \int_z^t \frac{1}{(s-z)^{\frac{2q}{q-1}}} ds dt.
\]

\[
\leq \frac{q}{(1+\delta)^{q+1} \max_{[z,z+\epsilon]} K(r)} |Y_1 - Y_2| \int_0^r \frac{1}{t^{q-1}} \int_z^t \frac{1}{(s-z)^{\frac{2q}{q-1}}} ds dt.
\]
Since $K(z) \neq 0$ and $K$ is continuous then $\max_{|z| \leq \epsilon} \frac{K(r)}{K(z)} \rightarrow 1$ as $\epsilon \rightarrow 0$. Also since $\delta > 0$ and $q < 1$ we see that for $\epsilon > 0$ sufficiently small then $\frac{q}{(1+\delta)^{1-q}} \max_{|z| \leq \epsilon} \frac{K(r)}{K(z)} \leq d \leq 1$.

For the second part of the integral since $g_1$ is locally Lipschitz at $W$ near 0 then:

$$\left| g_1 \left( (s - z)^{\frac{2}{p+1}} CY_1 \right) - g_1 \left( (s - z)^{\frac{2}{p+1}} CY_2 \right) \right| \leq L |s - z|^{\frac{2}{p+1}} C$$

so substituting into the second part of (78) gives:

$$-1 \left( \frac{1}{C(r-z)^{\frac{2}{p+1}}} \right) \int_z^1 \int_0^{r} s^{N-1} K(s) \left( g_1 \left( (s - z)^{\frac{2}{p+1}} CY_1 \right) \right)$$

$$- g_1 \left( (s - z)^{\frac{2}{p+1}} CY_2 \right) \right| ds \ dt.$$}

$$\leq \frac{|Y_1 - Y_2| CL}{C(r-z)^{\frac{2}{p+1}}} \max_{|z| \leq \epsilon} K(r) \int_z^1 \int_0^{r} s^{N-1} K(s) \left( |Y_1 - Y_2| L \max_{|z| \leq \epsilon} K(r) \right)$$

$$K(r) (r-z)^2 \leq \frac{1-d}{2} |Y_1 - Y_2|.$$}

$$\text{so that } L \max_{|z| \leq \epsilon} K(r) = 0 \text{ we can choose } \epsilon \text{ small enough}$$

$$\text{so combining these two part we get}$$

$$|TY_1(r) - TY_2|/|Y_1 - Y_2| \leq \frac{1+d}{2}$$

Thus $T$ is a contraction mapping if $0 < \frac{1+d}{2} < 1$ is sufficiently small, so there is a unique solution $Y \in B$ to $T(Y) = Y$ on $[z, z+\epsilon]$. Then $u_\epsilon(r) = -(r-z)^{\frac{2}{p+1}} W(r)$ is a solution of (4)–(5) on $[z-\epsilon, z+\epsilon]$ for some $\epsilon > 0$.

**Lemma 2:** The energy equation $E(r)$ is strictly decreasing.

**Proof:** From (12) we know that $E'(r) \leq 0$ so $E(r)$ is non-increasing. Suppose by way of contradiction that $E$ is not strictly decreasing then there are $r_1, r_2$ with $r_1 < r_2$ such that $E(r_1) = E(r_2)$ so $E(r)$ is constant on $[r_1, r_2]$ so $E'(r) \equiv 0$ on $[r_1, r_2]$ so $U_\epsilon'(r) \equiv 0$ on $[r_1, r_2]$ then by the uniqueness of solution of initial value problem $u_\epsilon \equiv 0$ on $[r, \infty]$ but $u_\epsilon'(R) = a > 0$ contradiction so $E$ must be strictly decreasing. This proofs lemma 2.

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**References**


Existence Solutions for a Singular Nonlinear Problem...


وجودية الحلول لمسألة متفردة غيرخطية مع الشروط الحدودية لدرجتة في المجال الخارجي

مجد علي 1، جوزيف اية 2

1 قسم الرياضيات، كلية العلوم، جامعة كركوك، كركوك، العراق.
2 قسم الرياضيات، جامعة شمال تكساس، دينتون، تكساس، الولايات المتحدة.

المؤلف: mageedali@uokirkuk.edu.iq

الخلاصة

لقد أثبت هذا البحث وجود الحلول التي تحل المعادلة التفاضلية الجزئية غير الخطية. لقد تم تطوير دراسة للأنظمة الديناميكية على الجزء الخارجي من النقطة المركزية في الأصل في 

$$ R > 0 $$

مع نصف القطر 0، مع شروط حدود من النوع الأول عندما 

$$ u = 0 $$

على الحدود، و (x) تقترب من 0، عندما 

$$ u = 0 $$

الدالة المتفردة في التالية، حيث الدالة (x) هي المتفردة البيضية المحلية عند الصفر، وتمono الدالة بشكل فائق عندما تقترب من النقطة. من خلال إدخال مقياس مختلف لتوضيح سلوك الدالة المتفردة بالقرب من المركز وفي النقطة، وت obs .، N > 2. نلاحظ أن الدالة تملك الدالة المتفردة في النقطة صغيرة مع 0 < q < 1. الدالة المتفردة لـ 

$$ u^q $$

عندما 

$$ u = 0 $$

الدالة تملك الدالة المتفردة في النقطة كبيرة مع 1 > q. بالإضافة إلى ذلك، 

$$ u $$

عندما 

$$ u = 0 $$

الدالة تملك الدالة المتفردة في النقطة الثانية وغيرها من التفاصيل لإثبات الوجودية.

الكلمات الدالة: المجالات الخارجة، المتفردة، النافذة، الوجودية.

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