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On the Butterfly Catastrophe Model and Stability of Finite Periodic Solutions for Some Non-Linear Differential Equations.

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Abstract

In this work, we find the results for the folded part projection of the butterfly catastrophe model onto the control space, using methods from catastrophe theory to obtain stability and the catastrophic behavior of finite periodic solutions for some non-linear differential equations. Finally, we have shown that a saddle-node bifurcation, which can be classified as a butterfly mutation, accompanies butterfly surface folding.

1. Introduction:

Studying dynamical systems with two-dimensional phase space, a limit cycle is a closed trajectory in phase space that has the property that it contains at least one other trajectory spirals either as the time approaches infinity or negative infinity. The Limit cycle is an isolated closed orbit in a system. Which is stable (or attractive) if all neighboring trajectories get close to it. If not, we say it is unstable. Catastrophe theory can explain several characteristics of the phenomena of discontinuous jumping which are very difficult to explain with the help equations.

The Butterfly type Catastrophe Model is developed for stability analysis by graphing the Butterfly-Model for non-linear differential equations, the bifurcation set, or the projection of the folding part of the butterfly catastrophe model (in [1] the projection of the folding part of the cusp catastrophe model) onto the control space is always accompanied with the saddle-node bifurcation. In catastrophe theory, rock blast failure patterns plays a very primary role in theoretical analysis and practical applications. Studying catastrophic problems such as equilibrium points, catastrophic manifold, capacitance, jump phenomena... etc.

has been of great interest for a long time due to its increasing applications in physical, biological, and social sciences. Some writers, such as K. D. Arrow Smith and K. L. Taha [2], L. Cesari[3], P. Hartman [4], K. L. Hale [5], W. Hirsch and S. Smile [6], C. Hayashi [7], E. J. Marsden and M. McCracken [8]. M.N. Mohammad [9], M.N. Murad Kaki[1],[10], W. D. Jordon and P. Smith [11], E. C. Zeeman [12] they made their valuable contributions to the study of some aspects (points of equilibrium, catastrophic model, limit cycles, periodic solutions, stability (instability), phenomena associated with the forced oscillations) of problems.

The aim of this work is to find limit cycles for the averaged system of the non-linear differential equations and their stabilities and semi-stabilities for the Butterfly catastrophe with six-degree non-linear differential equation, where [1] used Cusp catastrophe with a four-degree non-linear differential equation.

2. Systems rise from non-linear differential equation NLDE:

NLDE considered here from the model:

$$y'' = -w_0^2 y + \alpha f(x,y,y'), \quad ('= d \setminus dx) \tag{1}$$

Where α is the $\varepsilon-$ parameter and f is of period $\frac{2\pi}{\omega}$ concerning x, the linear form of Eq. (1) is not interesting because the catastrophic behavior appears only in the foregoing non-linear differential equation, and then we are proceeding to get the approximate solution of (1) for this purpose:

$$Let \quad y' = v, \tag{2}$$

Now equations (1) & (2), yields

$$v' = -w_0^2 y + \alpha f(x,y,y') \tag{3}$$

In order the set of equations:

$$\begin{aligned} y &= a(x)\sin(wx) + b(x)\cos(wx) \\ v &= w[a(x)\cos(wx) - b(x)\sin(wx)] \end{aligned} \tag{4}$$

must be a solution of equations (2) and (3), the conditions below are met [1]:

$$a'\sin(\omega x) - b'\cos(\omega x) = 0 \tag{5}$$

$$a'\cos(\omega x) - b'\sin(\omega x) = \frac{\alpha}{\omega}[\beta y + f(x,y,y')] \tag{6}$$

$$\alpha\beta = \omega^2 - \omega_0^2 \tag{7}$$

Hence Eqs. (5), (6), and (7) gives the dependent system below:

$$\begin{aligned} a' &= \frac{\alpha}{\omega}\{\beta y + f(x,y,y')\}\cos(\omega x) \\ b' &= -\frac{\alpha}{\omega}\{\beta y + f(x,y,y')\sin(\omega x)\} \end{aligned} \tag{8}$$

Integrating Eqs. (8) concerning x, for $0 < x < 2\pi/\omega$, we get :

$$\begin{aligned} a' &= \beta b + \mu a - \{\chi_2 ar^2 + \chi_4 ar^4 + \ldots + \chi_{2n} ar^{2n}\} \\ b' &= -\beta a + \mu b - \{\chi_2 br^2 + \chi_4 br^4 + \ldots + \chi_{2n} br^{2n}\} - B \end{aligned} \tag{9}$$

Whereas μ, β, B and $\chi_2, \chi_4, \ldots, \chi_2 n$ are the real parameters and $r = \sqrt{a^2+b^2}$ is the Amplitude. Which is the average system we want and which we get from the general form (1).

3. Catastrophic Manifold CM:

Since the stationary points of Eqs. (9) comes true if $a' = b' = 0$, so, equating to zero on the right-hand side of this system to zero and after some simplifications, we get:

$$[\mu r - (\chi_2 r^3 + \chi_4 r^5 + \ldots + \chi_{2n} r^{2n+1})]^2 + \beta^2 r^2 - B^2 = 0, \tag{10}$$

Where we used polar coordinate transformations:
$a = r\cos\phi, b = r\sin\varphi$. Putting $\zeta = r^2$ and if we perform the appropriate change of coordinates, Eq. (10) can be reduced to the standard form of some types of catastrophes, as well we may find some standard form of (10) for CM.

$$\zeta^m + u_1\zeta^{m-2} + u_2\zeta^{m-3} + + u_{m-1} = 0$$

This is our desired equation, where $m = 2n+1$. We define a function F' so that, we can find the non-linear dynamic model as follows after integrating concerning ζ:

$$F'(\zeta) = -(\zeta^m + u_1\zeta^{m-2} + u_2\zeta^{m-3} + \ldots + u_{m-1}) \tag{11}$$

The following is the canonical form for the potential function:

$$F(\zeta, u_1, u_2, \ldots) = \frac{1}{m+1}\zeta^{m+1} + \frac{u_1}{m-1}\zeta^{m-1} + \ldots + u_{m-1}\zeta \tag{12}$$

Resulting from the averaged system (10), where if $n = 2$, implies that $m = 5$ and F is the potential function of the butterfly-type catastrophe, which can be represented as:

$$F(\zeta, u_1, u_2, u_3, u_4) = \frac{1}{6}\zeta^6 + \frac{1}{4}u_1\zeta^4 + \frac{1}{3}u_2\zeta^3 + \frac{1}{2}u_3\zeta^2 + u_4\zeta \tag{13}$$

The stationary points of F are provided by

$$\frac{\partial F}{\partial \zeta} = \zeta^5 + u_1\zeta^3 + u_2\zeta^2 + u_3\zeta + u_4 = 0 \tag{14}$$

We take F and ζ also to be functions of the control variables as well, in this case,u_1, u_2, u_3, u_4 .The non-linear dynamic model is considered as:

$$F(\zeta) = -(\zeta^6 + u_1\zeta^4 + u_2\zeta^3 + u_3\zeta^2 + u_4\zeta) \tag{14a}$$

Also, let us look into the Lipsanos function of this dynamic. Construct a function: $F(\zeta, u_1, u_2, u_3, u_4) = \frac{1}{6}\zeta^6 + \frac{1}{4}u_1\zeta^4 + \frac{1}{3}u_2\zeta^3 + \frac{1}{2}u_3\zeta^2 + u_4\zeta$, via the butterfly catastrophe [13]. Someone saw that (14 a) is a Lyapunov function with:

$$\frac{dF}{dt} = -\left(\zeta^5 + u_1\zeta^3 + u_2\zeta^2 + u_3\zeta + u_4\right)^2 < 0 \Leftrightarrow \zeta^5 + u_1\zeta^3 + u_2\zeta^2 + u_3\zeta + u_4 \neq 0 \tag{15}$$

Therefore, in this section, the non-linear dynamical solution (14 a) is asymptotically stable. For the cusp catastrophe see [1], for which, if $n = 1$, then $m = 3$ in Eq. (10). The condition of three limit cycles is:

$$\Delta = 4u_1^3 + 27u_2^2 < 0 \tag{16}$$

The region's boundary (for one limit cycle or three) is defined as:

$$4u_1^3 + 27u_2^2 = 0 \tag{17}$$

Furthermore, we have the following propositions:
Proposition 3.1 There are two asymptotically stable solutions and two unstable solutions for any non-linear dynamical systems that arise from NLDE when $\Delta < 0$ in Eq. (16).
Proposition 3.2 The occurrence of the folding of Butterfly Catastrophe is almost always accompanied by saddle-node bifurcation.

4. Conclusion:

We have shown that the saddle-node bifurcation can be classified as a Butterfly type catastrophe and we have investigated asymptotically stable solutions and unstable solutions for any non-linear dynamical systems that arise from NLDE. We have also shown that the occurrence of the folding of Butterfly Catastrophe accompanies by saddle-node bifurcation.

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حول نموذج كارثة الفراشة واستقرارية الحلول الدورية المحددة لبعض المعادلات التفاضلية غير الخطية

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الخلاصة

في هذا العمل المتواضع وجدنا نتائج إسقاط الجزء المطوي من نموذج كارثة الفراشة على فضاء التحكم باستخدام اساليب وطرق من نظرية الكارثة لإيجاد الإستقرارية والسلوك الكارثي للحلول الدورية المحدودة لقسم من المعادلات التفاضلية اللاخطية. وبالتالي، بينا أن تشعب عجرة السرج، والتي يمكن تصنيفها على أنها طفرة فراشة، مترافقة مع طي لسطح الفراشة. وهو ما يعني انه دائما ما يكون إسقاط الجزء القابل للطي من نموذج كارثة الفراشة على فضاء التحكم يرافقه تشعب في عجرة السرج.

الكلمات الدالة: نموذج كارثة الفراشة؛ كارثة من نمط الفراشة؛ المعادلات التفاضلية اللاخطية ؛ الدارات الغائية.

التمويل: لايوجد.

بيان توفر البيانات: جميع البيانات الداعمة لنتائج الدراسة المقدمة يمكن طلبها من المؤلف المسؤول.

اقرارات:

تضارب المصالح: يقر المؤلفون أنه ليس لديهم تضارب في المصالح.

الموافقة الأخلاقية: لم يتم نشر المخطوطة أو تقديمها لمجلة أخرى، كما أنها ليست قيد المراجعة.