Kirkuk University Journal-Scientific Studies Vol. 18, Iss. 1, p. 1-15, 2023. doi:10.32894/kujss.2023.136740.1084

A New Version of Cubic Rank Transmuted Gumbel Distribution

Doaa Abd K. ELhertaniy*

Department of Mathematics, university of the Holy Quran and Taseel of Science, Sudan.

***Corresponding author** : dalhirtani@gmail.com

Article Information

Article Type:
Research Article

Keywords:
Moments; Gumbel Distribution; simulation study; order statistics; Entropy.

History:
Received: 03 November 2022.
Accepted: 15 January 2023.
Published: 31 March 2023.

Citation: Doaa Abd K. ELhertaniy, A New Version of Cubic Rank Transmuted Gumbel Distribution, Kirkuk University Journal - Scientific Studies, 18(1), P. 1-15, 2023, https://doi.org/10.32894/kujss.2023.136740.1084

Abstract

A Cubic Rank Transmuted Gumbel distribution (CTGD) in this article is a new generalization of the Gumbel distribution based on a cubic ranking transmutation map. Examined are the cubic transmuted Gumbel model's fundamental statistical properties, such as its hazard rate function, moment-generating function, moments, characteristic function, quantile function, entropy, and order statistics. Finally, the usefulness and applicability of the CTGD using two real data sets about waiting time at a bank is described and Wheaton River flood, and the fit has been compared with Gumbel distribution (GD) and transmuted Gumbel distribution (TGD). The results show that the proposed model provides a superior fit than transmuted Gumbel distributions and Gumbel distributions.

1. Introduction:

The Gumbel distribution is named after Emil Julius Gumbel (1891-1966), who described the distribution in his original writings. The Gumbel distribution is a special illustration of the generalized extreme value distribution (also known as the Fisher-Tippett distribution). It is sometimes referred to as the double exponential distribution and the log-Weibull distribution [1]. Perhaps the most frequently used statistical distribution for engineering challenges is the Gumbel distribution. It is sometimes referred to as the type I extreme value distribution. Flood frequency analysis, network engineering, nuclear engineering, offshore engineering, riskbased engineering, space engineering, software reliability engineering, structural engineering, and wind engineering are a few of its latest application areas in engineering.

Over fifty applications are included in a recent book by Kotz and Nadarajah [2] that describes this distribution and includes information on accelerated life tests, earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, wind speeds, and track race records (to mention just a few). It is one of four EVDs that are frequently used. The other three are the Generalized Extreme Value Distribution, the Weibull Distribution, and Frechet Distribution.

The Gumbel, Frechet, and Weibull families, commonly known as type I, type II, and type III extreme value distributions, have been combined into a single family of continuous probability distributions called the generalized extreme value (GEV) distribution. The probability density function (PDF) and the cumulative distribution function (CDF) for Gumbel distribution are defined as follow:

$$g(X;\mu,\sigma)=\frac{1}{\sigma}e^{-(z+e^{-z})} \tag{1}$$

where

$$z=(\frac{x-\mu}{\sigma}),x\in R,\sigma,\mu>0$$

and

$$G(X;\mu,\sigma)=e^{\left(-e^{-(\frac{x-\mu}{\sigma})}\right)},x\in R,\ \sigma,\ \mu>0 \tag{2}$$

Some extensions of the Gumbel distribution have previously been proposed. The Beta Gumbel distribution, Nadarajah et al.[3], the Exponentiated Gumbel distribution as a generalization of the standard Gumbel distribution introduced by Nadarajah [4], and the Exponentiated Gumbel type-2 distribution, studied by Okorie et al.[5].

Transmuted Gumbel type-II distributton with applications in diverse fields of science by Ahmad et al. [6], presenting Transmuted exponentiated Gumbel distribution (TEGD) and its application to water quality data of Deka et al. [7], and transmuted Gumbel distribution (TGD) along with several mathematical properties has studied by Aryal and Tsokos [8] using quadratic rank transmutation. Quadratic rank transmuted distribution has been proposed by Shaw and Buckley [9]. A random variable X is said to have a quadratic rank transmuted distribution if its cumulative distribution function is given by:

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2, \ |\lambda| \leq 1 \quad (3)$$

Differentiating (3) with respect to x, it gives the probability density function (pdf) of the quadratic rank transmuted distribution as:

$$f(x) = g(x)[\ (1+\lambda) - 2\lambda G(x)], \ |\lambda| \leq 1 \quad (4)$$

where G(x) and g(x) are the cdf and pdf respectively of the base distribution. It is very important observe that at λ = 0, we have the base original distribution. The family of quadratic transmuted distributions shown in (3) expands any baseline distribution G(x), increasing its applicability. Recently, Rahman et al. [10] introduced the cubic transmuted family of distributions. A random variable X is said to have cubic transmuted distribution with parameter λ if its cumulative distribution function (cdf) is given by:

$$F(x) = (1-\lambda)\ G(x) + 3\lambda[G(x)]^2 - \ 2\lambda[G(x)]^3 \quad (5)$$

with corresponding pdf

$$f(x) = g(x)[(1-\lambda) + 6\lambda G(x) - 6\lambda\ G^2(x)], \ x \in R \quad (6)$$

where $\lambda \in [-1,1]$.

This paper is organized as follows, in Section 2 defining the cubic transmuted Gumbel distribution. Statistical properties have been discussed such as the shapes of the density and hazard rate functions, quantile function, moments and moment-generating function, Characteristic Function, and cumulant generating function in Section 3. Entropy was studied in Section 4, and order statistics in Section 5. Section 6, we address the parameters of the CRTG distribution via maximum likelihood method.

A simulation study is carried out in Section 7 to assess performance of suggested maximum likelihood estimators. An application of the CTGD to two real data sets for the purpose of illustration is conducted in section 8. Finally, in Section 9, some conclusions are declared.

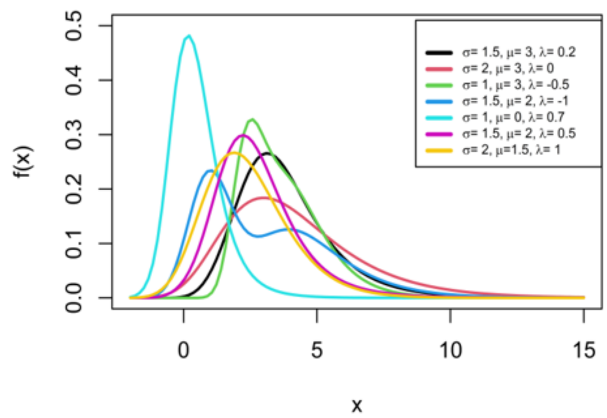

Figure 1. Plots of pdf plots of the CTG distribution.

2. A new version cubic transmuted Gumbel distribution:

The cubic rank transmuted Gumbel distribution is defined as follows: The CDF of a cubic rank transmuted Gumbel distribution is obtained by using (2) in (5).

$$F(x) = (1-\lambda)e^{(-e^{-(\frac{x-\mu}{\sigma})})} + 3\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2 - 2\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3 \quad (7)$$

where x ∈ R, $\mu > 0$, and $\sigma > 0$, are a location and scale parameters respectively, $\lambda \in [-1,1]$ is shape parameter.

It is very important note observe that at value $\lambda = 0$, the cubic rank transmuted Gumbel distribution reduce to Gumbel distribution according to the transmutation map. The probability density function (pdf) of a cubic rank transmuted Gumbel distribution is given by:

$$f(x) = \frac{1}{\sigma}e^{-(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})}[(1-\lambda)+6\lambda e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 6\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2] \quad (8)$$

where x ∈ R, $\mu,\sigma > 0$, λ ∈ [-1,1] Figure 1 and Figure 2, show different selected values of the model parameters λ, μ and σ for the pdf and cdf of the cubic rank transmuted Gumbel distribution.

3. Statistical Properties:

3.1 Shapes of the Density and Hazard Rate Functions:

The reliability function of the cdf of distribution is defined by $R(x) = 1 - F(x)$. For the cubic rank transmuted Gum-

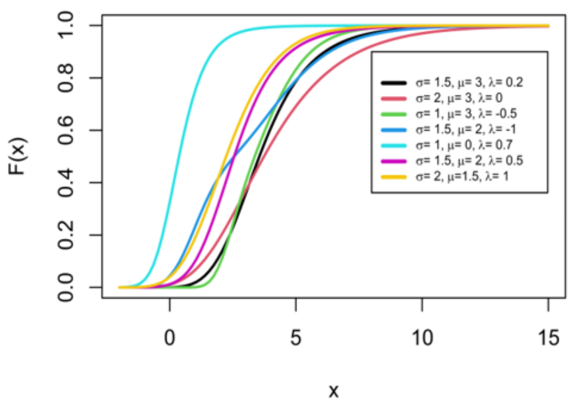

Figure 2. Plots of cdf of the CTG distribution.

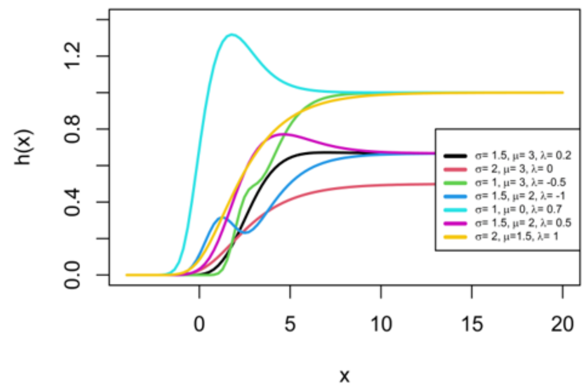

Figure 3. Plots of hazard function of the CTGD.

bel (CTG) distribution, the reliability function is given as:

$$R(X)=1-[(1-\lambda)e^{(-e^{-(\frac{x-\mu}{\sigma})})}+3\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2 -2\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3] \quad (9)$$

The hazard rate function can be written as the ratio of the pdf f (x) and the reliability function $R(x)=1-F(x)$. That is:

$$h(x)=\frac{f(x)}{R(x)},$$

then we can find the hazard rate function of GTF distribution by (8) and (9):

$$h(x)=\frac{\frac{1}{\sigma}e^{-(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})}[(1-\lambda)+6\lambda e^{(-e^{-(\frac{x-\mu}{\sigma})})}-6\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]}{1-[(1-\lambda)e^{(-e^{-(\frac{x-\mu}{\sigma})})}+3\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2-2\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3]} \quad (10)$$

The cumulative hazard function is defined by:

$$H(x)=-lnR(x),$$

so the cumulative hazard function of the CTG distribution is:

$$H(x)=-\ln\left\{1-[(1-\lambda)e^{(-e^{-(\frac{x-\mu}{\sigma})})}+3\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2 -2\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3]\right\} \quad (11)$$

The reverse hazard function is:

$$r(x)=\frac{f(x)}{F(x)} \quad (12)$$

Using (12), we can write the reverse hazard function of CTG distribution as

$$r(x)=\frac{\frac{1}{\sigma}e^{(-\frac{x-\mu}{\sigma})}[(1-\lambda)+6\lambda e^{(-e^{-(\frac{x-\mu}{\sigma})})}-6\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]}{(1-\lambda)+3\lambda[e^{(-e^{-\frac{x-\mu}{\sigma}})}]-2\lambda[e^{(-e^{-\frac{x-\mu}{\sigma}})}]^2} \quad (13)$$

Figure 3, show the Hazard function of the cubic rank transmuted Gumbel distribution for different values of parameters $\lambda,\ \mu$ and σ.

3.2 Quantile Function:

Here we will compute the quantile function of the cubic rank transmuted Gumbel probability distribution.

Theorem 3.1 Let X be random variable from the cubic rank transmuted Gumbel probability distribution with parameters $\sigma>0$, $\mu>0$ and $-1\leq\lambda\leq1$. Then the quantile function of X, is given by:

$$x_q=\mu-\sigma\log(-\log B(q,\lambda)) \quad (14)$$

Proof. To calculate the quantile function of the cubic rank transmuted Gumbel probability distribution, we substitute x by x_q and F(x) by q in (7) to get the equation

$$q=(1-\lambda)\ e^{(-e^{-(\frac{x_q-\mu}{\sigma})})}+3\lambda[e^{(-e^{-(\frac{x_q-\mu}{\sigma})})}]^2 -2\lambda[e^{(-e^{-(\frac{x_q-\mu}{\sigma})})}]^3 \quad (15)$$

Then, we solve the equation (15) for x_q. So, let $y=e^{(-e^{-(\frac{x_q-\mu}{\sigma})})}$ Thus, (15) becomes

$$q=(1-\lambda)y+3\lambda y^2-2\lambda y^3$$

and hence,

$$2\lambda y^3-3\lambda y^2-(1-\lambda)y+q=0 \quad (16)$$

let a = 2 λ, b = -3 λ, c = $(-1+\lambda)$ and d = q, then the equation (16) becomes

$ay^3+by^2+cy+d=0$

Then,

$$y=-\frac{b}{3a}-\frac{\sqrt[3]{2}\xi_1}{3a\left(\xi_2+\sqrt{4\xi_1^3+\xi_2^2}\right)^{\frac{1}{3}}}+\frac{\left(\xi_2+\sqrt{4\xi_1^3+\xi_2^2}\right)^{\frac{1}{3}}}{3\sqrt[3]{2}} \quad (17)$$

where $\xi_1=-b^2+3ac$, ξ_2 = - $2b^3$ + $9abc$ - $27ab^2d$, and $d=q$.

Now, let the function B(q, λ)be defined by:

$$B(q,\lambda)=-\frac{b}{3a}-\frac{\sqrt[3]{2}\xi_1}{3a(\xi_2+\sqrt{4\xi_1^3+\xi_2^2})^{\frac{1}{3}}}+\frac{(\xi_2+\sqrt{4\xi_1^3+\xi_2^2})^{\frac{1}{3}}}{3\sqrt[3]{2}}$$

Hence,

$$y=e^{(-e^{-(\frac{x_q-\mu}{\sigma})})}=B(q,\lambda) \quad (18)$$

Take natural Logarithm to both sides to get:

$$-e^{-(\frac{x_q-\mu}{\sigma})}=\log B(q,\lambda)$$

Then, we have the equation

$$X_q=\mu-\sigma\log(-\log B(q,\lambda))$$

put q = 0.25, 0.50, and 0.75 in (14) to obtained the first quartile, median and third quartile respectively.
Quantiles for selected parameter values for the CTG distribution are shown in Table 1.

Table 1. Inclusion and Exclusion Criteria for Related Publications.

	(σ, μ, λ)				
u	(0.5, 0.5, -0.5)	(1.5, 3, 0)	(5, 4, 0.5)	(2, 3, -1)	(0.1, 0.5, 1)
0.1	0.0151	1.7490	0.6894	0.8609	0.4511
0.2	0.1883	2.2862	2.3242	1.4955	0.4779
0.3	0.3415	2.7216	3.5730	2.0722	0.4987
0.4	0.5024	3.1311	4.7050	2.7521	0.5178
0.5	0.6833	3.5498	5.8326	3.7330	0.5367
0.6	0.8867	4.0076	7.0439	4.8873	0.5567
0.7	1.1152	4.5464	8.4552	5.9569	0.5795
0.8	1.3912	5.2499	10.2981	7.1174	0.6083
0.9	1.8003	6.3756	13.3111	8.7726	0.6524

3.3 Moments and Moment-generating function:

3.3.1 Moments function:

Theorem 3.2 Let X $\sim$ CTGD(μ, σ, λ). Then the rth moment of is given by:

$$E(x^r)=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sigma^j\mu^{r-j}\left[(1-\lambda)\frac{\partial^j}{\partial\alpha^j}\Gamma(\alpha)+6\lambda\frac{\partial^j}{\partial\alpha^j}(2^{-\alpha}\Gamma(\alpha))-6\lambda\frac{\partial^j}{\partial\alpha^j}(3^{-\alpha}\Gamma(\alpha))\right]|\alpha=1 \quad (19)$$

Proof. The rth moment is given by

$$E(x^r)=\int_{-\infty}^{\infty}x^rf(x)dx \quad (20)$$

Substituting from (8) in to (20),

$$E(x^r)=\int_{\infty}^{0}x^r\frac{1}{\sigma}e^{-(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})}[(1-\lambda)+6\lambda e^{(-e^{-(\frac{x-\mu}{\sigma})})}-6\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]dx \quad (21)$$

Using the transformation $y=e^{-(\frac{x-\mu}{\sigma})}$
Then dy = $\frac{-e^{-(\frac{x-\mu}{\sigma})}}{\sigma}dx$, x = μ - $\sigma\log y$
With substitution by this transformation in (21) then:

$$E(x^r)=\int_{\infty}^{0}[\mu-\sigma\log y]^r\frac{1}{\sigma}ye^{-y}[(1-\lambda)+6\lambda e^{-y}-6\lambda e^{-2y}]\frac{-\sigma}{y}dy$$
$$=-\int_{\infty}^{0}[\mu-\sigma\log y]^re^{-y}[(1-\lambda)+6\lambda e^{-y}-6\lambda e^{-2y}]dy$$
$$=\int_{0}^{\infty}[\mu-\sigma\log\ y]^re^{-y}[(1-\lambda)+6\lambda e^{-y}-6\lambda e^{-2y}]dy$$

Now, we calculate $[\mu-\sigma\log y]^r$ using binomial

$$=\int_0^{\infty}\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sigma^j\ \mu^{r-j}[\log y]^je^{-y}[(1-\lambda)+6\lambda e^{-y}-6\lambda e^{-y}]dy$$
$$=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sigma^j\mu^{r-j}\int_0^{\infty}[(1-\lambda)[\log y]^je^{-y}+6\lambda[\log y]^je^{-2y}-6\lambda[\log y]^je^{-3y}]dy$$

using Gamma integration:

$$\int_0^{\infty}t^{\alpha-1}[\log y]^re^{-y}dy=\frac{\partial^j}{\partial a^j}\int_0^{\infty}t^{\alpha-1}e^{-y}=\frac{\partial^j}{\partial\alpha^j}\Gamma(\alpha)$$

$$E(x)^r = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \sigma^j \mu^{r-j} \Bigg[(1-\lambda) \frac{\partial^j}{\partial \alpha^j} \Gamma(\alpha) + 6\lambda \frac{\partial^j}{\partial a^j} (2^{-\alpha}\Gamma(\alpha)) - 6\lambda \frac{\partial^j}{\partial a^j} (3^{-\alpha}\Gamma(\alpha)) \Bigg] |\alpha = 1$$

The mean and variance can be easily obtained by using r = 1, 2 in Eq. (19) such that:

$$\Gamma^{(0)}(1) = 1,\ \Gamma^{(1)}(1) = -\gamma \ and \ \Gamma^{(2)}(1) = \gamma^2 + \frac{\pi^2}{6}$$

where $\gamma \approx 0.5772$ is the Euler Mascheroni constant [11].

3.3.2 Moment Generating Function

Theorem 3.3 The moment generating function $M_x(t)$ of a random variable X $\sim$ CTGD (μ, σ) is given by:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sigma^k \mu^{r-k} \Bigg[(1-\lambda) \frac{\partial^k}{\partial \alpha^k} \Gamma(\alpha) + 6\lambda \frac{\partial^k}{\partial \alpha^k} (2^{-\alpha}\Gamma(\alpha)) - 6\lambda \frac{\partial^k}{\partial \alpha^k} (3^{-\alpha}\Gamma(\alpha)) \Bigg] |\alpha = 1 \quad (22)$$

Proof. We know that:

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx$$

Using series expansion of e^{tx}

$$M_x(t) = \sum_{r=0}^{n} \frac{t^r}{r!} \int_0^{\infty} x^r f(x) dx = \sum_{r=0}^{n} \frac{t^r}{r!} E(x^r)$$

Then

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sigma^k \mu^{r-k} \Bigg[(1-\lambda) \frac{\partial^k}{\partial \alpha^k} \Gamma(\alpha) + 6\lambda \frac{\partial^k}{\partial \alpha^k} (2^{-\alpha}\Gamma(\alpha)) - 6\lambda \frac{\partial^k}{\partial \alpha^k} (3^{-\alpha}\Gamma(\alpha)) \Bigg] |\alpha = 1$$

Some moments for selected parameters values in order (σ, μ, λ): (V1): (1, 0, 0.5), (V2): (1.5, 3, 1), (V3): (2, 1.5, 0.7), (V4): (2, 3, -1) and (V5): (0.5, 0.5, -0.5) are given in Table 2, where CV, SD, CK, and CS represent the coefficient of variation, standard deviation, kurtosis, and skewness, respectively and 3D plots of kurtosis and skewness for the CTG distribution are given in Figure 4 and Figure 5. We observe that:

- When the parameter λ is fix, the kurtosis and skewness of CTGD decrease as σ decreases.

- When we fix the parameters μ, the skewness and kurtosis of CTGD decrease as σ decreases.

Table 2. Moments for Selected Parameters for CTG Distribution.

	V1	V2	V3	V4	V5
E(X)	0.18413872	0.0012319141	0.08432721	0.05696914	0.26488573
E(X^2)	0.11711591	0.0010517312	0.05923564	0.04077740	0.16895885
E(X^3)	0.08504845	0.0009192204	0.04576221	0.03183581	0.12298493
E(X^4)	0.06647332	0.0008171177	0.03731107	0.02613115	0.09640258
E(X^5)	0.05443340	0.0007358141	0.03150427	0.02216664	0.07918527
E(X^6)	0.04602701	0.0006694404	0.02726512	0.01924883	0.06715645
SD	0.03983899	0.0006141780	0.02403304	0.01701063	0.05828806
CV	0.03510047	0.0005674239	0.02148672	0.01523900	0.05148284
CS	0.03135916	0.0005273367	0.01942862	0.01380168	0.04609728
CK	0.02833227	0.0004925753	0.01773050	0.01261213	0.04172973

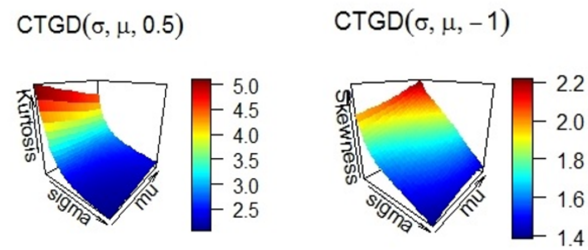

Figure 4. Plots of kurtosis and skewness for the CTG distribution at various parameter values.

3.4 Characteristic Function:

The cubic transmuted Gumbel distribution's characteristic function theorem is stated as follows:

Theorem 3.4 Assume that the random variable X have the CTGD (μ, σ, λ), then characteristic function, $\Phi_x(t)$, is:

$$\Phi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sigma^k \mu^{r-k} \Bigg[(1-\lambda) \frac{\partial^k}{\partial \alpha^k} \Gamma(\alpha) + 6\lambda \frac{\partial^k}{\partial \alpha^k} (2^{-\alpha}\Gamma(\alpha)) - 6\lambda \frac{\partial^k}{\partial \alpha^k} (3^{-\alpha}\Gamma(\alpha)) \Bigg] |\alpha = 1 \quad (23)$$

Where i = $\sqrt{-1}$ and t $\in \Re$

3.5 Cumulant Generating Function:

The cumulant generating function is defined by:

$$K_x(t) = \log_e M_x(t)$$

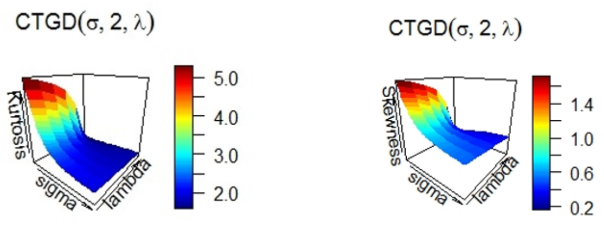

Figure 5. Plots of kurtosis and skewness for the CTG distribution at various parameter values.

Cumulant generating function of cubic rank transmuted Gumbel distribution is given by:

$$K_x(t) = \log_e \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sigma^k \mu^{r-k} \Bigg[(1-\lambda) \frac{\partial^k}{\partial \alpha^k} \Gamma(\alpha) + 6\lambda \frac{\partial^k}{\partial \alpha^k} (2^{-\alpha}\Gamma(\alpha)) - 6\lambda \frac{\partial^k}{\partial \alpha^k} (3^{-\alpha}\Gamma(\alpha)) \Bigg] |\alpha = 1 \tag{24}$$

4. Entropy:

4.1 Rényi Entropy:

If X is a non-negative continuous random variable with pdf $f(x)$, then the Renyi entropy of δ order (See Renyi [12]) of X is defined as,

$$H_\delta(x) = \frac{1}{1-\delta} \log \int_0^\infty [f(x)]^\delta dx, \forall \delta > 0, (\delta \neq 1) \tag{25}$$

Theorem 4.1 The Rényi entropy of a random variable X $\sim$ CTGD (μ, σ), with $\lambda \neq 1$ and $\lambda \neq 0$ is given by:

$$H_\delta(x) = \frac{1}{1-\delta} \log \left[\sum_{j=0}^{\infty} \sum_{k=0}^{j} c(j,k,\delta) \lambda^j (1-\lambda)^{\delta-j} \frac{-1}{\sigma^{\delta-1}} \frac{\Gamma(\delta)}{(\delta+j+k)^\delta} \right]$$

Proof. Assume X has the pdf in (8). Then, can compute.

$$[f(x)]^\delta = \frac{1}{\sigma^\delta} e^{-\delta(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})} \Bigg[(1-\lambda) + 6\lambda e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 6\lambda [e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2 \Bigg]^\delta \tag{26}$$

By the general binomial expansion, we have:

$$\left[(1-\lambda) + 6\lambda e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 6\lambda [e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2 \right]^\delta = \sum_{j=0}^{\infty} \binom{\delta}{j} (1-\lambda)^{\delta-j} \left[6\lambda e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 6\lambda e^{-2(e^{-(\frac{x-\mu}{\sigma})})} \right]^j \tag{27}$$

by the Binomial Theorem,

$$\begin{aligned} &\left[6\lambda e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 6\lambda e^{-2(e^{-(\frac{x-\mu}{\sigma})})} \right]^j \\ &= \sum_{k=0}^{j} \binom{j}{k} \left[6\lambda e^{(-e^{-(\frac{x-\mu}{\sigma})})} \right]^{j-k} \left[-6\lambda e^{-2(e^{-(\frac{x-\mu}{\sigma})})} \right]^k \\ &= \sum_{k=0}^{j} \binom{j}{k} 6^{j-k} (\lambda)^{j-k} e^{-(j-k)(e^{-(\frac{x-\mu}{\sigma})})} (-6)^k \lambda^k e^{-2k(e^{-(\frac{x-\mu}{\sigma})})} \\ &= \sum_{k=0}^{j} \binom{j}{k} (-1)^k 6^j \lambda^j e^{-(j+k)(e^{-(\frac{x-\mu}{\sigma})})} \end{aligned} \tag{28}$$

Substitute from(28) in (27), to get:

$$\begin{aligned} &\left[(1-\lambda) + 6\lambda e^{(-e^{-(\frac{x-\mu}{\sigma})})} - 6\lambda [e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2 \right]^\delta \\ &= \sum_{j=0}^{\infty} \binom{\delta}{j} (1-\lambda)^{\delta-j} \left[\sum_{k=0}^{j} \binom{j}{k} (-1)^k 6^j \lambda^j e^{-(j+k)(e^{-(\frac{x-\mu}{\sigma})})} \right] \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{\delta}{j} \binom{j}{k} (-1)^k 6^j (1-\lambda)^{\delta-j} \lambda^j e^{-(j+k)(e^{-(\frac{x-\mu}{\sigma})})} \end{aligned} \tag{29}$$

Now, substitute from(29) in (26), to get

$$[f(x)]^\delta = \frac{1}{\sigma^\delta} e^{-\delta(\frac{x-\mu}{\sigma})+e^{-(\frac{x-\mu}{\sigma})})} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{\delta}{j} \binom{j}{k} (-1)^k 6^j (1-\lambda)^{\delta-j} \lambda^j e^{-(j+k)(e^{-(\frac{x-\mu}{\sigma})})}$$

Let c(j, k, δ) = $\binom{\delta}{j} \binom{j}{k} (-1)^k 6^j$, then:

$$[f(x)]^\delta = \frac{1}{\sigma^\delta} \sum_{j=0}^{\infty} \sum_{k=0}^{j} c(j,k,\delta) \lambda^j (1-\lambda)^{\delta-j} e^{[-(\delta+j+k)(e^{-(\frac{x-\mu}{\sigma})})+(-\delta\frac{(x-\mu)}{\sigma})]} \tag{30}$$

To find $H_\delta(x)$, we substitute from (30) in (25).

$$H_\delta(x) = \frac{1}{1-\delta} \log\Bigg[\sum_{j=0}^{\infty}\sum_{k=0}^{j} c(j,k,\delta)\frac{1}{\sigma^\delta}\lambda^j(1-\lambda)^{\delta-j}\times \int_0^\infty e^{[-(\delta+j+k)(e^{-(\frac{x-\mu}{\sigma})})+(-\delta(\frac{x-\mu}{\sigma})]}dx\Bigg] \quad (31)$$

We can evaluate the integration by using the transformation

$$\int_0^\infty e^{[-(\delta+j+k)(e^{-(\frac{x-\mu}{\sigma})})+(-\delta\frac{(x-\mu)}{\sigma})]}dx = \int_0^\infty (e^{-\frac{(x-\mu)}{\sigma}})^\delta \times e^{-(\delta+j+k)(e^{-(\frac{x-\mu}{\sigma})})}dx \quad (32)$$

let y = $e^{-(\frac{x-\mu}{\sigma})}$, and $\delta > 1$, then $dy = \frac{-e^{-(\frac{x-\mu}{\sigma})}}{\sigma}dx$ and 0 < y < ∞

incorporating gamma integration and transformation in (32) we get,

$$-\sigma\int_0^\infty y^{\delta-1}e^{-(\delta+j+k)y}dy = -\sigma\frac{\Gamma(\delta)}{(\delta+j+k)^\delta} \quad (33)$$

After solving the integral, we get the Rényi entropy of the CTGD (μ, σ) by substitute from (33) in (31).

$$H_\delta(x) = \frac{1}{1-\delta} log\Bigg[\sum_{j=0}^{\infty}\sum_{k=0}^{j} c(j,k,\delta)\lambda^j(1-\lambda)^{\delta-j}\frac{-1}{\sigma^{\delta-1}} \frac{\Gamma(\delta)}{(\delta+j+k)^\delta}\Bigg]$$

4.2 q-Entropy:

Havrda and Charvat [13] established the concept of q-entropy. It is the Shannon entropy's one-parameter generalization. According to Aman Ullah [14], the q-entropy is:

$$I_H(q) = \frac{1}{1-q}[1-\int_0^\infty f(x)^q dx], \quad where\ q>0,\ and\ q\neq 1 \quad (34)$$

Theorem 4.2 The q-entropy of a random variable X ∼ CTGD (μ, σ), with $\lambda \neq 1$ and $\lambda \neq 0$ is given by:

$$I_H(q) = \frac{1}{1-q}\Bigg[1-\sum_{j=0}^{\infty}\sum_{k=0}^{j} c(j,k,q)\lambda^j(1-\lambda)^{q-j}\frac{-1}{\sigma^{q-1}} \frac{\Gamma(q)}{(q+j+k)^q}\Bigg]$$

Proof. To find $I_H(q)$, we substitute (30) in (34).

4.3 Shannon Entropy:

In a non-negative continuous random variable X with pdf f(x), the Shannons entropy [15] is defined as:

$$H(f) = E[-log\ f(x)] = -\int_0^\infty f(x)log(f(x))dx \quad (35)$$

The Expansion of the Logarithm function will be used below (Taylor series at 1),

$$log(x) = \sum_{m=0}^{\infty}(-1)^{m-1}\frac{(x-1)^m}{m},\ |x|<1 \quad (36)$$

The Shannon entropy of a random variable X ∼ CTGD (μ, σ), with $\lambda \neq 1$ and $\lambda \neq 0$ is given by:

$$H(f) = \sum_{m=1}^{\infty}\sum_{n=0}^{m}\sum_{j=0}^{n+1}\sum_{k=0}^{j}(-1)^n\binom{m}{n}\frac{1}{m}c(j,k,n+1)\frac{-1}{\sigma^n} \times\lambda^j(1-\lambda)^{n+1-j}\frac{\Gamma(n+1)}{(n+1+j+k)^{n+1}} \quad (37)$$

Proof. Using the logarithm function's expansion (36)

$$\log(f(x)) = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{(f(x)-1)^m}{m},$$

and by Binomial Theorem,

$$= \sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{m}\left\{\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}f^n(x)\right\}$$

$$\log(f(x)) = \sum_{m=1}^{\infty}\sum_{n=0}^{m}(-1)^{n+1}\binom{m}{n}\frac{1}{m}f^n(x) \quad (38)$$

To compute the Shannon's entropy of X, substitute from (38) in (35).

$$H(f) = -\int_0^\infty f(x)\log(f(x))dx = -\int_0^\infty f(x)\sum_{m=1}^{\infty}\sum_{n=0}^{m}(-1)^{n+1} \times\binom{m}{n}\frac{1}{m}f^n(x)dx$$

$$H(f) = \int_0^\infty\sum_{m=1}^{\infty}\sum_{n=0}^{m}(-1)^n\binom{m}{n}\frac{1}{m}f^{n+1}(x)dx \quad (39)$$

substituting from (30) in (39), to get:

$$H(f)=\int_0^\infty\sum_{m=1}^{\infty}\sum_{n=0}^{m}(-1)^n\binom{m}{n}\frac{1}{m}\sum_{j=0}^{n+1}\sum_{k=0}^{j}c(j,k,n+1)\frac{1}{\sigma^{n+1}}$$
$$\times\lambda^j(1-\lambda)^{n+1-j}e^{[-(n+1+j+k)(e^{-(\frac{x-\mu}{\sigma})})+(-(n+1)\frac{(x-\mu)}{\sigma})]}dx$$
$$H(f)=\sum_{m=1}^{\infty}\sum_{n=0}^{m}\sum_{j=0}^{n+1}\sum_{k=0}^{j}(-1)^n\binom{m}{n}\frac{1}{m}c(j,k,n+1)\frac{1}{\sigma^{n+1}}\times\lambda^j$$
$$(1-\lambda)^{n+1-j}\int_0^\infty e^{[-(n+1+j+k)(e^{-(\frac{x-\mu}{\sigma})})+(-(n+1)\frac{(x-\mu)}{\sigma})]}dx$$

Now, we use (33) to fined the value of the integration, so,

$$H(f)=\sum_{m=1}^{\infty}\sum_{n=0}^{m}\sum_{j=0}^{n+1}\sum_{k=0}^{j}(-1)^n\binom{m}{n}\frac{1}{m}c(j,k,n+1)\frac{-1}{\sigma^{n}}$$
$$\times\lambda^j(1-\lambda)^{n+1-j}\frac{\Gamma(n+1)}{(n+1+j+k)^{n+1}}$$

5. Order Statistics:

Let $X_1, X_2, ..., X_n$ be a random sample of size k from the CTG distribution with parameters $\mu>0$, $\sigma>0$, and $-1\leqslant\lambda\leqslant1$ From (Casella and Berger [16]), the pdf of the k th order statistics is obtain by:

$$f_{x_{(k)}}(x)=k\binom{n}{k}f(x)[F(x)]^{k-1}[1-F(x)]^{n-k} \tag{40}$$

Let X_k be the k th order statistic from X $\sim$ CTGD (μ, σ) with $\lambda\neq1$ and $\lambda\neq0$. Then pdf of the k th order statistic is given by:

$$\begin{aligned}f_{x_{(k)}}(x)=&k\binom{n}{k}f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}\\&=k\binom{n}{k}f(x)[F(x)]^{k-1}\left[\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}[F(x)]^j\right]\\&=\sum_{j=0}^{n-k}(-1)^jk\binom{n}{k}\binom{n-k}{j}f(x)[F(x)]^{k+j-1}\end{aligned}$$

$$f_{x_{(k)}}(x)=\sum_{j=0}^{n-k}(-1)^jk\binom{n}{k}\binom{n-k}{j}\frac{1}{\sigma}e^{-\left(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})}\right)}[1-\lambda$$
$$+6\lambda e^{-(e^{-(\frac{x-\mu}{\sigma})})}-6\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2]\times\left[(1-\lambda)\right.$$
$$\left.e^{(-e^{-(\frac{x-\mu}{\sigma})})}+3\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^2-2\lambda[e^{(-e^{-(\frac{x-\mu}{\sigma})})}]^3\right]^{k+j-1}$$

6. Maximum Likelihood Estimation (MLE):

Assume $X_1, X_2, ..., X_n$ be random sample of size n from CTGD (μ, σ) hen the likelihood function can be written as:

$$l(\mu,\sigma,\lambda)=\prod_{i=1}^{n}f(x_i)=$$
$$\prod_{i=1}^{n}\left[\frac{1}{\sigma}e^{-(\frac{x_i-\mu}{\sigma}+e^{-(\frac{x_i-\mu}{\sigma})})}[(1-\lambda)+6\lambda e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}\right.$$
$$\left.-6\lambda[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2]\right]$$

Then

$$l(\mu,\sigma,\lambda)=\frac{1}{\sigma^n}e^{-\Sigma_{i=1}^{n}(\frac{x_i-\mu}{\sigma}+e^{-(\frac{x_i-\mu}{\sigma})})}\Pi_{i=1}^{n}\left[[(1-\lambda)+6\lambda\right.$$
$$\left.e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-6\lambda[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2\right] \tag{41}$$

Then, the Log likelihood function of a vector of parameters given as,

$$\log l(\mu,\sigma,\lambda)=\log\prod_{i=1}^{n}f(x_i)=-n\log\sigma+\log$$
$$(e^{-\Sigma_{i=1}^{n}(\frac{x_i-\mu}{\sigma}+e^{-(\frac{x_i-\mu}{\sigma})})})+\sum_{i=1}^{n}\log$$
$$\left[(1-\lambda)+6\lambda e^{-(e^{-(\frac{x_i-\mu}{\sigma})})}-6\lambda\right.$$
$$\left.[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2\right]$$

Then

$$log\ l(\mu,\sigma,\lambda)=-nlog(\sigma)-\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma}+e^{-(\frac{x_i-\mu}{\sigma})})$$
$$+\Sigma_{i=1}^{n}\log\left[(1-\lambda)+6\lambda e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-6\lambda[e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}]^2\right]$$

Differentiate w.r.t parameters μ, σ, and λ we have.

$$\frac{\partial\log l}{\partial\mu}=\frac{n}{\sigma}-\frac{1}{\sigma}\sum_{i=1}^{n}e^{-(\frac{x_i-\mu}{\sigma})}+\frac{1}{\sigma}\sum_{i=1}^{n}$$
$$\frac{-6\lambda e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}e^{-(\frac{x_i-\mu}{\sigma})}+12\lambda e^{(-2e^{-(\frac{x_i-\mu}{\sigma})})}e^{-(\frac{x_i-\mu}{\sigma})}}{(1-\lambda)+6\lambda e^{(-e^{-(\frac{x_i-\mu}{\sigma})})}-6\lambda e^{(-2e^{-(\frac{x_i-\mu}{\sigma})})}} \tag{42}$$

$$\frac{\partial \log l}{\partial \mu} = \frac{-n}{\sigma} + \sum_{i=1}^{n} \frac{x_i - \mu}{\sigma^2} - \sum_{i=1}^{n} e^{-(\frac{x_i-\mu}{\sigma})} \frac{x_i - \mu}{\sigma^2} + \sum_{i=1}^{n} \frac{6\lambda e^{(-e^{-(\frac{x_i-\mu}{\sigma})})} e^{-(\frac{x_i-\mu}{\sigma})} \frac{x_i-\mu}{\sigma^2} - 12\lambda e^{(-2e^{-(\frac{x_i-\mu}{\sigma})})} e^{-(\frac{x_i-\mu}{\sigma})} \frac{x_i-\mu}{\sigma^2}}{(1-\lambda)+6\lambda e^{(-e^{-(\frac{x_i-\mu}{\sigma})})} - 6\lambda e^{(-2e^{-(\frac{x_i-\mu}{\sigma})})}} \tag{43}$$

$$\frac{\partial \log l}{\partial \sigma} = \sum_{i=1}^{n} \frac{-1 - 6e^{\left(-2e^{-(\frac{x_i-\mu}{\sigma})}\right)} + 6e^{\left(-e^{-(\frac{x_i-\mu}{\sigma})}\right)}}{(1-\lambda) + 6\lambda e^{(-e^{-(\frac{x_i-\mu}{\sigma})})} - 6\lambda e^{(-2e^{-(\frac{x_i-\mu}{\sigma})})}} \tag{44}$$

By setting the above nonlinear equations to zero, we can use the maximum likelihood method to estimate the unknown parameters.

$$\begin{aligned} \frac{\partial \log l(\mu,\sigma,\lambda)}{\partial \mu} &= 0 \\ \frac{\partial \log l(\mu,\sigma,\lambda)}{\partial \sigma} &= 0 \\ \frac{\partial \log l(\mu,\sigma,\lambda)}{\partial \lambda} &= 0 \end{aligned} \tag{45}$$

and solving them simultaneously. Therefore, in order to acquire the numerical solution to the nonlinear equations, can utilize statistical software. using quasi-Newton procedure,or computer packages/ softwares such as R, SAS, Ox, MATLAB and MATHEMATICA, We can calculate the maximum likelihood estimators (MLEs) of parameters (μ, σ, λ).

7. Simulation Study:

In this section, simulation results are presented for different sample sizes of n = 100, 200, 350, 500 and 600 to check the consistency and accuracy of the maximum likelihood estimators (MLEs) for each CTG distribution parameter. The simulation was conducted N = 1000 times, and the root mean square errors (RMSEs), average bias (A Bias), and mean estimations were assessed. The mean estimations are displayed in Tables 3 and 4 together with the corresponding RSMEs and A Bias. The A Bias and RMSEs for the estimated parameter, say,θ are respectively given as:

A Bias $(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta$
and,
RMSE $(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{n} (\hat{\theta}_i - \theta)^2}{N}}$

Table 3. Monte Carlo simulation results from the CTG distribution.

		(3, 5, -0.6)			(1.5, 3, 1)		
Parameter	Size	Mean	RMSE	A Bias	Mean	RMSE	A Bias
σ	100	2.9683869	0.3110686	-0.031613	1.4124447	0.192761	-0.0875553
	200	2.9653400	0.1943333	-0.034660	1.4413629	0.145419	-0.0586371
	350	2.9641475	0.1468471	-0.035852	1.4566656	0.118423	-0.0433344
	500	2.9681908	0.1186902	-0.031809	1.4646765	0.097915	-0.0353235
	600	2.9689276	0.1092767	-0.031072	1.4720971	0.083501	-0.0279029
μ	100	5.0102457	0.3529291	0.0102457	3.0234538	0.115822	0.0234538
	200	5.0231378	0.2558181	0.0231378	3.0188759	0.087923	0.0188759
	350	5.0211866	0.1845014	0.0211866	3.0141480	0.065121	0.0141479
	500	5.0100060	0.1508559	0.0100059	3.0069940	0.052363	0.0069939
	600	5.0048706	0.1389350	0.0048706	3.0027601	0.047960	0.0027601
λ	100	-0.6095491	0.3137271	-0.009549	0.8807370	0.285499	-0.1192630
	200	-0.6248843	0.2207617	-0.024884	0.9219822	0.205312	-0.0780178
	350	-0.6364474	0.1655242	-0.036447	0.9390858	0.161835	-0.0609142
	500	-0.6270135	0.1357510	-0.027014	0.9512945	0.138257	-0.0487055
	600	-0.6267492	0.1190665	-0.026749	0.9621599	0.112930	-0.0378401

Table 4. Monte Carlo simulation results from the CTG distribution.

		(2, 3, 0.5)			(0.6, 0.7, -1)		
Parameter	Size	Mean	RMSE	Bias	Mean	RMSE	Bias
σ	100	1.9211129	0.334973	-0.07889	0.6136448	0.0530885	0.01364479
	200	1.9606604	0.279166	-0.03934	0.6109134	0.0332931	0.01091343
	350	1.9747647	0.234069	-0.02524	0.6097105	0.0256254	0.00971053
	500	1.9758268	0.202012	-0.02417	0.6084322	0.0211626	0.00843219
	600	1.9777069	0.190073	-0.02229	0.6082418	0.0195105	0.00824184
μ	100	3.0176148	0.199861	0.017615	0.7042277	0.0678061	0.00422769
	200	3.0145087	0.152499	0.014509	0.7053112	0.0494191	0.00531121
	350	3.0109260	0.113605	0.010926	0.7040583	0.0348269	0.00405834
	500	3.0024912	0.098721	0.0024912	0.7012063	0.0282066	0.00120633
	600	3.0000571	0.090058	0.0000571	0.7003267	0.0268318	0.00032672
λ	100	0.3537058	0.459299	-0.14629	-0.864626	0.2423701	0.13537376
	200	0.4147267	0.363055	-0.08527	-0.903963	0.1624097	0.09603691
	350	0.4392960	0.302085	-0.06070	-0.929262	0.1190584	0.07073831
	500	0.4479509	0.258816	-0.05205	-0.938991	0.0989909	0.06100867
	600	0.4540239	0.232968	-0.04598	-0.941779	0.0912471	0.05822062

From the results, we can verify that as the sample size n increases, the mean estimates tend to be closer to the true parameter values, whereas the RSMEs and A Bias decrease for all parameter values.

8. Applications:

In this section, the Cubic Rank Transmuted Gumble Distribution (CTGD) is applied on two data sets as follows:

Data Set 1: The data set consists of observations on the amount of time a customer waited (in minutes) before receiving service from a bank. Ghitany et al. [17] . The data is: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4. 8, 4.9, 4.9, 5, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7,10.9, 11, 11, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.3, 3.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19, 19.9, 20.6,21.3, 21.4, 21.9, 23, 27, 31.6, 33.1, 38.5.

Data Set 2: The second data set that was acquired [18] relates to the data are the exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada: 1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9 , 13, 12, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39, 0.3, 15, 11, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9, 1.7, 7, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64, 41.5, 2.5, 27.4, 1, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.

Using R software to execute model parameter estimations and goodness-of-fit tests, the CTG distribution was compared to other existing models. The goodness-of-fit measures used for model performance comparison are, (-2$\mathscr{L}(\theta)$): where is $\mathscr{L}(\theta)$ the maximum value of log-likelihood function, AIC (Akaike Information Criterion), AICc (Corrected Akaike Information Criterion) and BIC (Bayesian information Criterion). Graphical plots such fitted densities, empirical cdfs, hrfs plots, Kaplan-Meier, and TTT plots were used to examine the Model fit was also investigated. Furthermore, the Kolmogorov-Smirnov $(K-S)$ statistics and associated p-values were obtained along with the Cramer-von Mises (w^*) and Andersen-Darling (A^*) statistics. A very good fit of the model to the data is shown by reduced values for all three goodness-of-fit indicators. Large p-values also indicate a good fit for the model, which is another benefit. The new CTG distribution was compared to the Gumbel distribution (GD) , transmuted Gumbel distribution (TGD), with pdf

$$f_{TG} = \frac{1}{\sigma} e^{-(\frac{x-\mu}{\sigma}+e^{-(\frac{x-\mu}{\sigma})})}[(1+\lambda)-2\lambda e^{(-e^{-(\frac{x-\mu}{\sigma})})}]$$

Table 5. Descriptive Statistics of Data Set 1, 2.

Data	mean	Median	Skewness	kurtosis
Set1	9.89	8	1.479	5.7559
set2	12.204	9.5	1.504	6.3199

8.1 Data1:

In a bank, the waiting time (in minutes) before the customer received service For the data1, this subsection includes parameter estimates (standard errors in parenthesis), goodness-of-fit statistics, plots of the fitted densities, empirical cdf, hrf graphs, probability plots, Kaplan-Meier and TTT plots

The CTG distribution fit the data the best, as shown in Figure 6, Figure 7, and Figure 8. According to the fitted density, the CTG distribution can handle skewed data. In data1, the estimated variance-covariance matrix for the CTGD model is given b

$$\begin{bmatrix} 0.149 & 0.072 & 0.057 \\ 0.072 & 0.249 & -0.024 \\ 0.057 & -0.024 & 0.086 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\sigma \in [4.29 \mp 0.756]$, $\mu \in [7.18 \mp 0.979]$, and $\lambda \in$

$$[-0.715 \mp 0.755]$$

Table 6. MLE's of the parameters and respective SE's for various distributions for Data Set 1.

Distribution	Parameter	Estimate	SE
CTGD	μ	7.18082	0.38553
	σ	4.29210	0.49963
	λ	-0.71534	0.29316
TGD	μ	7.57178	1.06169
	σ	5.29926	0.64624
	λ	0.27170	0.31391
GD	μ	6.80551	0.52627
	σ	4.97704	0.41531

Table 7. MLE's of the parameters and respective SE's for various distributions for Data Set 1.

Model	-2L (θ)	AIC	BIC	AICC	W^*	A^*	K-S	P-value
CTGD	628.8966	634.89	642.65	635.15	0.0641	0.4218	0.0639	0.819
TGD	632.4556	638.46	646.21	638.71	0.0931	0.6016	0.0764	0.617
GD	633.1347	637.14	642.30	637.26	0.1018	0.6581	0.0788	0.578

In Table 7, we compare the CTG model with the TG, and G distributions. Its noted that the proposed model has the lowest values for the AIC, AICC, W^* and A^* statistics among all fitted models (except BIC for the GD), as well as the highest p value. So, the CTGD can be chosen as the best model among the competing distributions studied in this article.

Data2: Wheaton River flood data

For the data2, this subsection includes parameter estimates (standard errors in parenthesis), goodness-of-fit statistics, plots of the fitted densities, empirical cdf, hrf graphs, probability plots, Kaplan-Meier and TTT plots.

The CTG distribution fit the data the best, as shown in Figure 9, Figure 10, and Figure 11. According to the fitted density, the CTG distribution can handle skewed data. In data2, the estimated variance-covariance matrix for the CTGD model is given by

$$\begin{bmatrix} 0.506334 & 0.372909 & 0.11991 \\ 0.372909 & 0.733727 & 0.06072 \\ 0.119911 & 0.060722 & 0.08869 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

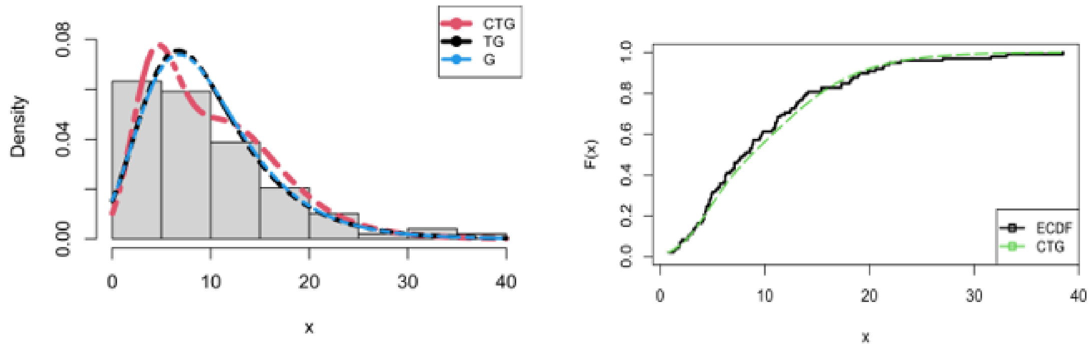

Figure 6. Fitted densities and empirical cdf plots for data1.

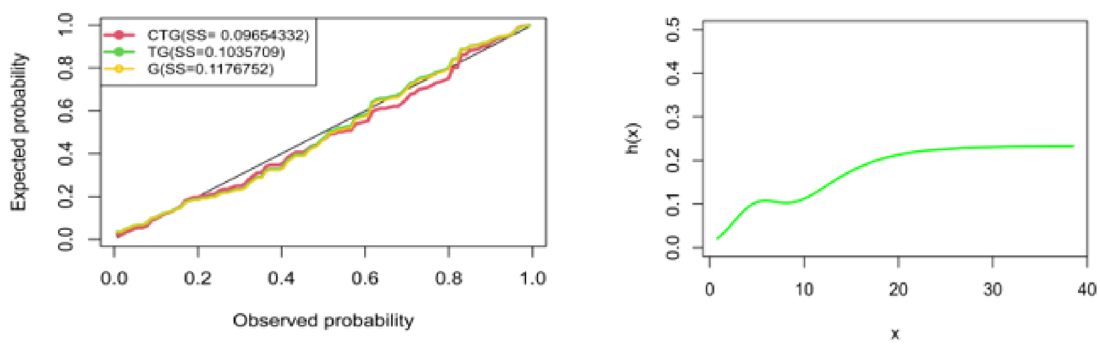

Figure 7. Probability plot and hrf plot for data1.

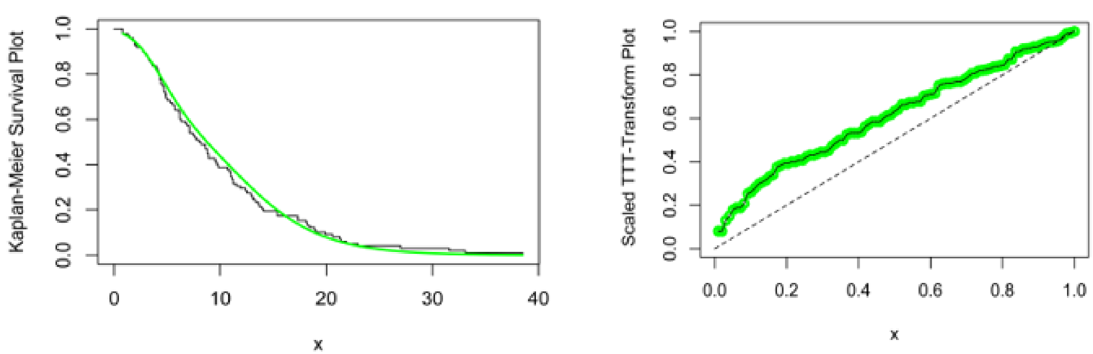

Figure 8. Kaplan-Meier and TTT plots for data1.

$\sigma \in [6.446 \mp 1.39], \mu \in [7.34 \mp 1.679],$ *and*
$\lambda \in [-1 \mp 0584]$

Table 8. MLE's of the parameters and respective SE's for various distributions for Data Set 2.

Distribution	Parameter	Estimate	SE
CTGD	μ	7.34024	0.85658
	σ	6.44583	0.71157
	λ	-1.00000	0.29782
TGD	μ	7.89765	1.70109
	σ	8.55889	1.08914
	λ	0.20997	0.29933
GD	μ	6.96837	1.00925
	σ	8.18927	0.81851

Table 9. Goodness-of fit statistics using the selection criteria values for Data Set 2.

Model	-2L (θ)	AIC	BIC	AICC	W^*	A^*	K-S	P-value
CTGD	526.3687	532.3687	539.1987	532.7216	0.18981	1.32803	0.13048	0.1722
TGD	538.4016	544.4016	551.2316	544.7545	0.27258	1.859597	0.1556	0.06123
GD	538.8729	542.8729	547.4262	543.0468	0.275508	1.875759	0.15777	0.0555

From the results in Table 9, CTG distribution performed better than any other model. It had the lowest values for $-2\log L$, AIC, AICC, BIC, W^*, A^*, and K-S, as well as the highest p-value when compared to competing models across for Wheaton River flood data. Also These plots indicate that the CTG distribution provides a better fit than others models considered for both data.

9. Concluding Remarks:

This article examines the cubic rank transmuted Gumbel (CTG) distribution, a novel generalized distribution. The distribution's hazard function, quantile function, moments, distribution of the order statistics, and entropy are among the structural aspects that are examined. The model parameters are estimated using a technique called maximum likelihood estimation. To investigate the performance of the CTG distribution, a Monte Carlo simulation analysis was carried out. The importance and potential of the CTG distribution is demonstrated by examples from two real life data sets.

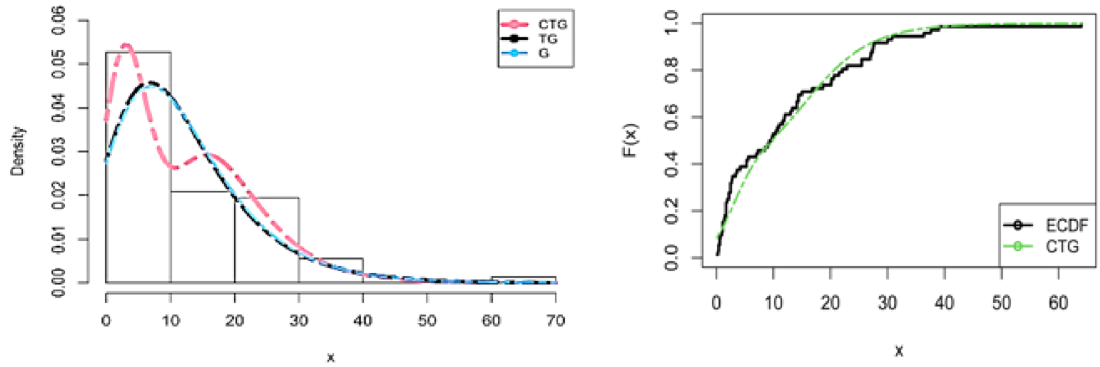

Figure 9. Fitted densities and empirical cdf plots for data2.

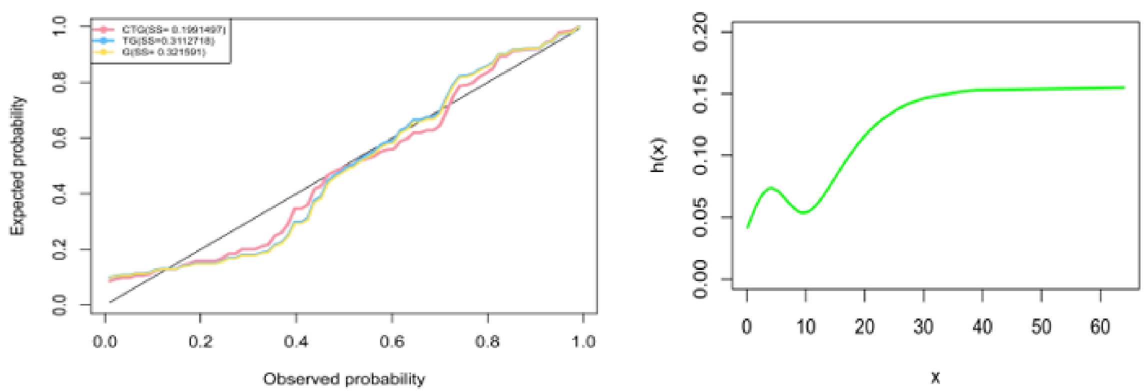

Figure 10. Probability plot and hrf plot for data2.

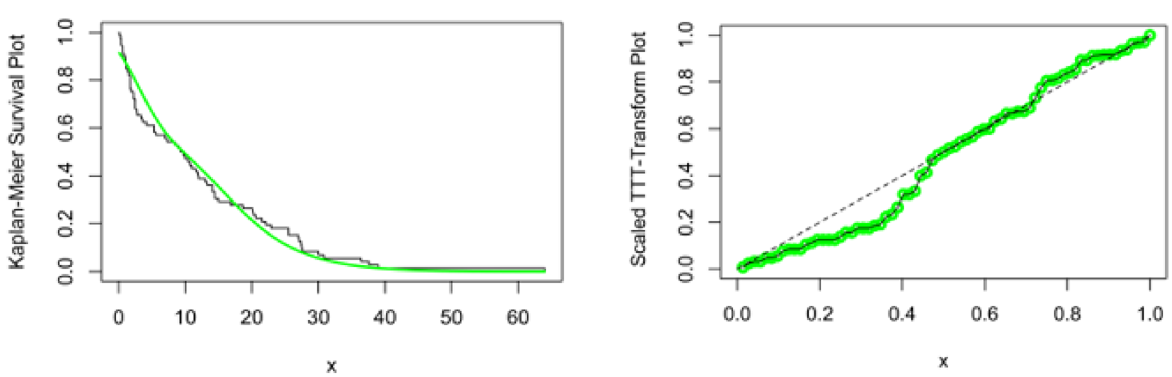

Figure 11. Kaplan-Meier and TTT plots for data2.

Funding: None.

Data Availability Statement: All of the data supporting the findings of the presented study are available from corresponding author on request.

<u>**Declarations:**</u>
Conflict of interest: The authors declare that they have no conflict of interest.

Ethical approval: The manuscript has not been published or submitted to another journal, nor is it under review.

References

[1] Les valeurs extremes des distributions statistiques. *Annales de l'institut Henri Poincaré*, 5(2), 1935.

[2] Kotz Samuel and Nadarajah Saralees. *Extreme value distributions: theory and applications*. world scientific, London, 2^{ed} edition, 2000.

[3] Saralees Nadarajah and Samuel Kotz. The beta gumbel distribution. *Mathematical Problems in engineering*, 2004: 323–332, 2004, doi:10.1155/S1024123X04403068.

[4] Saralees Nadarajah. The exponentiated gumbel distribution with climate application. environmetrics. *The official journal of the International Environmetrics Society*, 17: 13–23, 2006.

[5] Idika E. Okorie, Anthony Akpanta, and J. Ohakwe. The exponentiated gumbel type-2 distribution: properties and application. *International Journal of Mathematics and Mathematical Sciences*, 2016(2): 1–10, 2016, doi:10.1155/2016/5898356.

[6] S. Qurat ul Ain Ahmad Aijaz and Rajnee Tripathi. Transmuted gumbel type-ii distribution with applications in diverse fields of science. *Pakistan Journal Statistics*, 37: 429–446, 2021.

[7] Deepshikha Deka, Bhanita Das, and Bhupen K. Baruah. Transmuted exponentiated gumbel distribution (tegd) and its application to water quality data. *Pakistan Journal of Statistics and Operation Research*, 13(1): 115–126, 2017, doi:10.18187/pjsor.v13i1.1636.

[8] Gokarna R. Aryal and Chris P. Tsokos. On the transmuted extreme value distribution with application. *Nonlinear Analysis*, 71: e1401–e1407, 2009.

[9] William T. Shaw and Ian RC. Buckley. he alchemy of probability distributions: beyond gramcharlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *arXiv:0901.0434*, 2009, doi:10.48550/arXiv.0901.0434.

[10] Bander Al-Zahran Md. Mahabubur Rahman, Saman H Shahbaz, and Muhammad Qaiser Shahbaz. Cubic transmuted uniform distribution: An alternative to beta and kumaraswamy distributions. *European Journal of Pure and Applied Mathematics*, 12: 1106–1121, 2019, doi:10.29020/nybg.ejpam.v12i3.3410.

[11] Zhongfeng Sun and Huizeng Qin. Some results on the derivatives of the gamma and incomplete gamma function for non-positive integers. *IAENG International Journal of Applied Mathematics*, 47(3): 265–270, 2017.

[12] Alfréd Rényi. On measures of entropy and information. *In Proceedings of the* 4^{th} *Berkeley symposium on mathematics, statistics and probability*, 1: 547–561, 1961.

[13] Jan Havrda and František Charvát. Quantification method of classification processes. concept of structural a-entropy. *Kybernetika*, 3(1): 30–35, 1967.

[14] Aman Ullah. Entropy, divergence and distance measures with econometric applications. *Journal of Statistical Planning and Inference*, 49(1): 137–162, 1996, doi:10.1016/0378-3758(95)00034-8.

[15] Claude E. Shannon. A mathematical theory of communication. *The Bell system technical journal*, 27(3): 379–423, 1948, doi:10.1002/j.1538-7305.1948.tb00917.x.

[16] G. Casella and L. Berger. *Statistical inference*. Cengage Learning, USA, 2^{ed} edition, 2021.

[17] Mohamed E.Ghitany, Atieh Barbra, and Saralees Nadarajah. Lindley distribution and its application, mathematics and computers in simulation. *Mathematics and computers in simulation*, 78: 493–506, 2008, doi:10.1016/j.matcom.2007.06.007.

[18] Faton Merovci and Lukan Puka. Transmuted pareto distribution. *ProbStat Forum*, 7(1): 1–11, 2014.

توسيع جديد من الرتبة التكعيبية لتوزيع غامبل المحول

دعاء عبد الرحمن خليل الحرتاني *
قسم الرياضيات، جامعة القران الكريم وتأصيل العلوم، السودان.
* **الباحث المسؤول:** *dalhirtani@gmail.com*

الخلاصة

توزيع غامبل المحول ذي الرتبة التكعيبية (*CTGD*) في هذا البحث هو تعميم جديد لتوزيع غامبل استناداً إلى خريطة تحويل الترتيب التكعيبي. تم فحص الخصائص الإحصائية الأساسية لنموذج غامبل المحول ذي الرتبة التكعيبية، مثل دالة معدل الخطر، ودالة توليد العزوم، والعزوم، والدالة المميزة، والدالة الكمية، والإنتروبيا، والإحصاءات المرتبة. أخيراً تم وصف فائدة وإمكانية تطبيق *CTGD* باستخدام مجموعتين حقيقيتين من البيانات، حول وقت الانتظار في أحد البنوك وفيضانات نهر ويثون، وقد تمت مقارنة الملاءمة مع توزيع غامبل و غامبل المحول. أظهرت النتائج أن النموذج المقترح، هو أفضل ملائمة من توزيعات غامبل المحول و غامبل.

الكلمات الدالة: العزوم؛ توزيع غامبل؛ دراسة المحاكاة ؛الاحصاءات المرتبة؛ إنتروبيا.

التمويل: لايوجد.

بيان توفر البيانات: جميع البيانات الداعمة لنتائج الدراسة المقدمة يمكن طلبها من المؤلف المسؤول.

اقرارات:
تضارب المصالح: يقر المؤلفون أنه ليس لديهم تضارب في المصالح.

الموافقة الأخلاقية: لم يتم نشر المخطوطة أو تقديمها لمجلة أخرى، كما أنها ليست قيد المراجعة.