Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 14, Issue 3, September 2019 , pp. (33-49)
ISSN: 1992-0849 (Print), 2616-6801 (Online)

Estimating Parameters for Extension of Burr Type X Distribution by Using Conjugate Gradient in Unconstrained Optimization

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Abstract

The Conjugate gradient method used to estimate the parameter of Marshall-Olkin Exponentiated Burr Type X distribution (MOEBX). The proposed distribution MOEBX based on the work by (Marshall–Olkin 1997). Several properties of the MOEBX distribution were investigated and studied such as quantile function, moments, moment generation function and order statistics. The estimation process by maximum likelihood estimation maybe an obstacle for statisticians, so used Conjugate Gradient method in unconstrained optimization to estimate parameters. It was employed for estimating the three parameters of the new distribution. The flexibility of the MOEBX was illustrated by using two real data sets. We compared with nested and no nested distributions and encouraging results were obtained using a real data set.

Keywords: Exponentiated Burr Type X, Marshall-Olkin, Conjugate Gradient, unconstrained Optimization, MLE.

DOI: http://doi.org/10.32894/kujss.2019.14.3.4

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Volume 14, Issue 3, September 2019 , pp. (33-49)
ISSN: 1992-0849 (Print), 2616-6801 (Online)

تقدير المعالم لتوزيع بور العاشر الموسع (Burr Type X) باستخدام التدرج المترافق في الامثلية اللامقيدة

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الملخص

تم استخدام طريقة التدرج المترافق لتقدير معالم توزيع مارشال اولكين – تعميم بور العاشر. يستند عمل التوزيع المقرح الى طريقة (مارشال اولكين –1997). العديد من خصائص التوزيع المقترح تمت دراستها كدالة الكمية والعزوم ودالة المولدة للعزوم والإحصاءات المرتبة. ان تقدير المعالم بطريقة الإمكان الأعظم قد يكون عملية صعبة (عائق) امام الاحصائيين، لذا وظفت طريقة التدرج المترافق الغير مقيدة في الامثلية لتقدير معلمات التوزيع الجديد، ولتوضيح مرونة التوزيع المقترح تم استخدام مجموعتين من البيانات الحقيقية. قورن التوزيع الجديد مع توزيعات فرعية وتوزيعات غير المتداخلة، وتم الحصول على نتائج مشجعة باستخدام مجموعة بيانات حقيقية.

الكلمات الدالة: توسيع بور العاشر، مارشال اولكين، التدرج المترافق، الامثلية اللامقيدة، متوسط مربعات الخطأ.

DOI: http://doi.org/10.32894/kujss.2019.14.3.4

Volume 14, Issue 3, September 2019 , pp. (33-49)
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1. Introduction:

Twelve distributions were introduced by [1] using differential equation approach; of this, Burr type XII and Burr Type X distributions have received adequate attention in the literature [2]. For instance, various extensions of the Burr type X (BX) distribution has been introduced in recent times, the two-parameter BX distribution [3], Beta BX distribution [4], Exponentiated Generalized BX distribution [2], Gamma BX distribution [5], Transmuted BX distribution [6] and several others are notable examples. The usefulness of these distributions have however been demonstrated using real life applications.

The interest of this research, a new method was presented of the parameter estimation by using the conjugate gradient method in unconstrained optimization. In additional to extend the Exponentiated BX distribution using the Marshall-Olkin's method [7] of generating new distributions because of its flexibility. Several other families of distributions are available in the literature, readers are referred to [8,9,10,11, and 12] for further details.

The remaining part of this paper is structured in the following manner; in section 2, the new model, Marshall-Olkin Exponentiated Burr X (MOEBX) distribution is derived including its statistical properties while real life applications are provided in section 3.

2. Marshall-Olkin Exponentiated Burr Type X (MOEBX) Distribution:

Suppose Y denote a random variable (R.V), the cumulative distribution (cdf) and the probability density functions (pdf) of the BX distribution are;

$$F(y,\varphi) = \left(1 - e^{-y^2}\right)^{\varphi} \tag{1a}$$

$$f(y,\varphi) = 2\,\varphi\, y\, e^{-y^2}(1 - e^{-y^2})^{\varphi-1} \tag{1b}$$

Now, following the work of [13], the cdf of the Exponentiated Burr X (EBX) distribution is obtained as;

$$W(y,\varphi,\Omega) = 1 - \left[1 - \left(1 - e^{-y^2}\right)^{\varphi}\right]^{\Omega}; \qquad y > 0, \varphi > 0, \Omega > 0 \tag{2}$$

The pdf of the EBX refers to;

$$w(y,\varphi,\Omega) = 2\Omega\,\varphi\, y e^{-y^2}\left(1 - e^{-y^2}\right)^{\varphi-1}\left[1 - \left(1 - e^{-y^2}\right)^{\varphi}\right]^{\Omega-1} \tag{3}$$

where φ and Ω are two shape parameters.

In 1997, Marshal and Olkin [7] introduced a method for adding one additional shape parameter as follows;

Let $\bar{W}(y) = 1 - W(y)$ denote the survival function for a R.V of Y,

$$\bar{G}(y) = \frac{\rho \bar{W}(y)}{W(y) + \rho \bar{W}(y)} \quad ; \quad -\infty < y < \infty \; ; \quad \rho > 0 \tag{4}$$

and the CDF of Marshall-Olkin G family is

$$G(y) = \frac{\bar{W}(y)}{W(y)+\rho\bar{W}(y)} \tag{5}$$

$$w(y) = \frac{dW(y)}{dy}$$

$$g(y) = \frac{\rho W(y)}{\left[1 - \bar{\rho}\bar{W}(y)\right]^2} \quad ; \quad \rho > 0 \tag{6}$$

where $\bar{\rho} = 1 - \rho$

By inserting Equation (3) into Equation (4), the cdf of the MOEBX distribution is obtained as;

$$\bar{G}(y,\varphi,\Omega,\rho) = \frac{\rho\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega}}{1-\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega} + \rho\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega}} \tag{7}$$

The corresponding pdf of the MOEBX is obtained as;

$$g(y,\varphi,\Omega,\rho) = \frac{2\varphi\Omega\rho y e^{-y^2}\left(1-e^{-y^2}\right)^{\varphi-1}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega-1}}{\left\{1-\bar{\rho}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega}\right\}^2} \tag{8}$$

The expression in Equation (8) can be re-written as;

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$$g(y,\varphi,\Omega,\rho) = 2\varphi\Omega\rho\, y e^{-y^2}\left(1-e^{-y^2}\right)^{\varphi-1}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega-1}\left\{1-\bar{\rho}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega}\right\}^{-2} \tag{9}$$

The plot for the pdf of MOEBX distribution is as illustrated in Fig. 1.

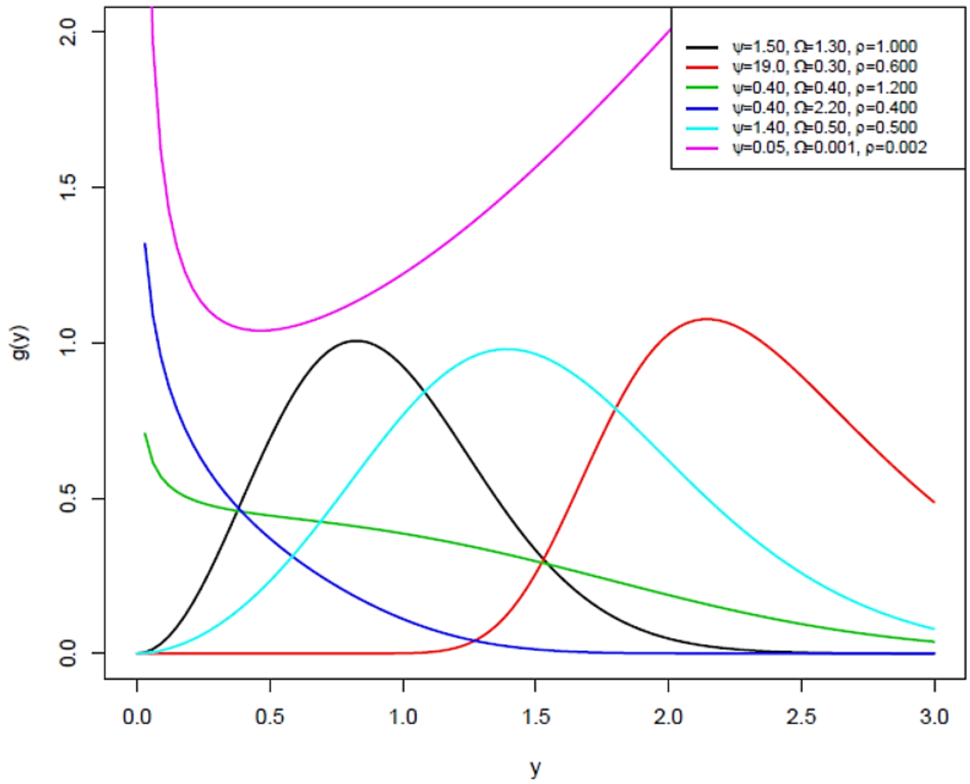

Fig. 1: PDF plot for the MOEBX distribution.

We observe from Fig. 1 that the MOEBX distribution exhibits decreasing, bathtub and unimodal shapes. Its clear that rhe new model shapes look like some distributions shaped such as Beta, Gamma, BX and Exponential by choosing different values of parameters.

2.1 Expansion for the pdf:

The pdf of the MOEBX distribution was obtained in Equation (9) as:

$$g(y,\varphi,\Omega,\rho) = 2\varphi\Omega\rho\, y e^{-y^2}\left(1-e^{-y^2}\right)^{\varphi-1}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega-1} \times \left\{1-\bar{\rho}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega}\right\}^{-2}$$

Using binomial expansion and for $0 < \rho < 1$, the expression $\left\{1-\bar{\rho}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega}\right\}^{-2}$ reduces to give:

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$$\left\{1-\bar{\rho}\left[1-\left(1-e^{-y^{2}}\right)^{\varphi}\right]^{\Omega}\right\}^{-2}=\sum_{i=0}^{\infty}(i+1)\left(\bar{\rho}\right)^{i}\left[1-\left(1-e^{-y^{2}}\right)^{\varphi}\right]^{i\Omega} \tag{10}$$

Substituting Equation (10) in Equation (9), we get:

$$g(y,\varphi,\Omega,\rho)=\sum_{i=0}^{\infty}2\varphi(i+1)\Omega\rho\left(\bar{\rho}\right)^{i}ye^{-y^{2}}\left(1-e^{-y^{2}}\right)^{\varphi-1}\left[1-\left(1-e^{-y^{2}}\right)^{\varphi}\right]^{(i+1)^{-1}\Omega} \tag{11}$$

The expression in Equation (11) can be re-written as:

$$g(y,\varphi,\Omega,\rho)=\sum_{i=0}^{\infty}2\varphi\Omega^{*}\eta_{i}ye^{-y^{2}}\left(1-e^{-y^{2}}\right)^{\varphi-1}\left[1-\left(1-e^{-y^{2}}\right)^{\varphi}\right]^{\Omega^{*}-1} \tag{12}$$

$$=\sum_{i=0}^{\infty}\eta_{i}w_{EBX}\left(y;\varphi,\Omega^{*}\right)$$

where $\eta_i = \rho\,\bar{\rho}^{\,i}$; $\quad \Omega^{*}=(i+1)\Omega$

w_{EBX} is the pdf of Exponentiated Burr X (EBX) distribution with parameters φ and Ω^{*}.

For $\rho>1$, we can use the same argument as in Equation (12), after some algebraic calculations we obtain:

$$g(y;\varphi,\Omega,\rho)=\sum_{i=0}^{\infty}\gamma_{i}f_{EBX}\left(y;\varphi,\Omega^{*}\right) \tag{13}$$

where; $\gamma_i=\dfrac{(-1)^{i}}{\rho(i+1)}\displaystyle\sum_{m=i}^{\infty}(m+1)\binom{m}{j}\left(1-\frac{1}{\rho}\right)^{m}$

Therefore, the pdf of the MOEBX distribution can be expressed as an infinite linear combination of Exponentiated Burr Type X pdf. Moreover, Equations (12) and (13) are used to find the mathematical properties such as the r-th moments of the MOEBX distribution.

2.2 Hazard and Survival Function:

Hazard function is obtained using:

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$$h(y) = \frac{g(y)}{G(y)}$$

Therefore, the hazard function of the MOEBX distribution is:

$$h(y,\varphi,\Omega,\rho) = 2\varphi\Omega y e^{-y^2}\left(1-e^{-y^2}\right)^{\varphi-1}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{-1}\left\{1-\bar{\rho}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega}\right\}^{-2} \tag{14}$$

The plot for the hazard function of MOEBX distribution is as illustrated in Fig. 2.

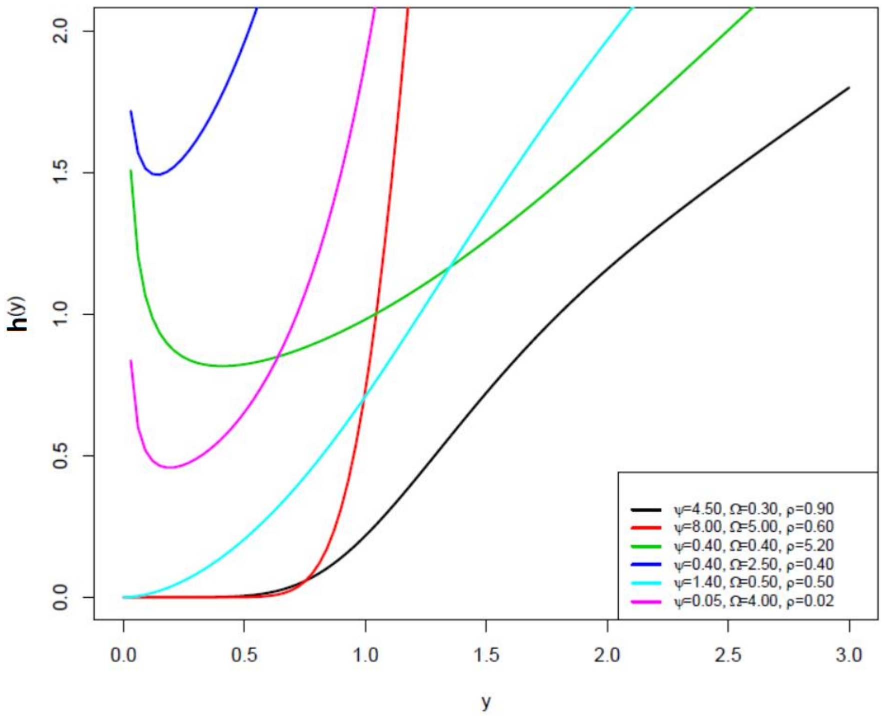

Fig. 2: Plot for the hazard function of MOEBX distribution.

The survival function (S.F) on the other hand is obtained from:

$$S(y) = 1 - G(y)$$

Therefore, we get the S.F function of the MOEBX distribution as:

$$S(y,\varphi,\Omega,\rho) = 1 - \frac{\rho\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega}}{1-\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega} + \rho\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega}} \tag{15}$$

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2.3 Quantile Function and Median:

The quantile function (Qf) which is otherwise known as the inverse cdf is obtained from:

$$Q(u) = G^{-1}(u)$$

Therefore, Qf for the MOEBX distribution is:

$$Q(u) = \left\{ -\log\left\{ 1 - \left[1 - \left(\frac{1-u}{1-u\bar{\rho}} \right)^{1/\Omega} \right]^{1/\varphi} \right\} \right\}^{1/2} \tag{16}$$

This indicates that random samples can be generated for the MOEBX distribution using eq (17) as follows:

$$y = \left[-log\left\{ 1 - \left(1 - \left(\frac{1-U}{1-U\bar{\rho}} \right)^{1/\Omega} \right)^{1/\varphi} \right\} \right]^{1/2} \tag{17}$$

where ‘U’ is uniformly distributed with parameters (0,1). The first and third quartiles can also be obtained when U=0.25 and U=0.75 in Equation (16) respectively.

The median can be written by using u=0.5 as follows:

$$Q(0.5) = \left[-log\left\{ 1 - \left(1 - \left(\frac{0.5}{1-0.5\bar{\rho}} \right)^{1/\Omega} \right)^{1/\varphi} \right\} \right]^{1/2}$$

2.4 Moments:

By using the r-th moment of EBX distribution which is:

$$\mu_r = \varphi\Omega \sum_{\rho, j=0}^{\infty} (-1)^{\rho+j} \binom{\Omega}{\rho} \binom{\varphi(\rho+1)}{j} \frac{\Gamma\left(\frac{r}{2}+1\right)}{(\rho+1)^{\frac{r}{2}+1}} \tag{18}$$

We obtain the r-th moment for the MOEBX distribution as:

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$$\mu_r = \sum_{i=0}^{\infty} \eta_i \varphi \sum_{l,j=0}^{\infty} \Omega^* (-1)^{l+j} \binom{\Omega^*}{l} \binom{\varphi(l+1)}{j} \frac{\Gamma\left(\frac{r}{2}+1\right)}{(l+1)^{\frac{r}{2}+1}} \tag{19}$$

where; $\Omega^* = (i+1)\Omega$, and $\eta_i = \rho\left(\bar{\rho}\right)^i$

The moment generating function (mgf) of the MOEBX is therefore given by;

$$M_y(Ee^{yt}) = \sum_{i,l,j,r=0}^{\infty} \frac{t^r \eta_i \varphi \Omega^* (-1)^{l+j}}{r!} \binom{\Omega^*}{l} \binom{\varphi(l+1)}{j} \frac{\Gamma\left(\frac{r}{2}+1\right)}{l+1^{\frac{r}{2}+1}} \tag{20}$$

2.5 Order Statistics:

If $y_1, y_2, \ldots, y_n$ are random samples from a cdf and pdf generated from the MOEBX distribution, the pdf of the *ith* order statistics of the MOEBX distribution is thus obtained as follows:

$$g_{i:n}(y) = \frac{g(y)}{B(i, n-i+1)} \sum_{j=0}^{n-1} (-1)^j \binom{n-j}{j} [G(y)]^{i+j-1}$$

Where $B(i, n-i+1)$is represented Beta distribution.

Since $\left[G(y)\right]^{i+j-1} = \left[\frac{G(y)}{1-\bar{\alpha}\bar{G}(y)}\right]^{i+j-1} = \left[\frac{1-\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega}}{1-\bar{\rho}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\Omega}}\right]^{i+j-1}$

Such that $\bar{\rho} = 1-\rho$

After some calculations, we have:

$$g_{i:n}(y) = \frac{g(y,\varphi,\Omega,\rho) \sum_{j=0}^{n-1} \sum_{l,r=0}^{\infty} (-1)^{l+j} \binom{n-j}{j} \binom{i+j-1}{l} \binom{i+j}{r} \rho^{-r} \left[1-(1-e^{-y^2})^{\varphi}\right]^{(l+r)\Omega}}{B(i,n-i+1)} \tag{21}$$

Using Equation (11), the expansion of $g(y,\varphi,\Omega,\rho)$ and substituting it in Equation (21) to reduce equation above as follows;

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$$g_{i;n}(y)=\frac{\sum_{s=1}^{\infty}\varphi(s-1)\Omega\rho y e^{-y^{2}}\left(1-e^{-y^{2}}\right)^{\varphi-1}}{B(i,n-i+1)}\times$$

$$\sum_{j=0}^{n-1}\sum_{l,r=0}^{\infty}(-1)^{l+j}\binom{n-i}{j}\binom{i+j-1}{l}\binom{i+j}{r}\times\rho\left(\bar{\rho}\right)^{r+s}\left[1-\left(1-e^{-y^{2}}\right)^{\varphi}\right]^{\Omega(r+l+s+1)-1}$$

So,

$$g_{i;n}(y)=\sum_{s=0}^{\infty}C_{s}\,g_{(r+l+s+1)}(y)$$

where;

$$C_{s}=\frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-1}\sum_{l,r=0}^{\infty}(-1)^{l+j}\binom{n-i}{j}\binom{i+j-1}{l}\binom{i+j}{r}\times\frac{\rho\left(\bar{\rho}\right)^{r+s}}{r+l}$$

and

$g_{(r+l+s+1)}$ denote the EBX density function with parameters $(\varphi,(r+l+s+1)\Omega)$.

3. Maximum Likelihood Estimation by Conjugate Gradient (CG) Method:

The parameters were estimated by using maximum likelihood function by CG method in unconstrained optimization (Fletcher–Reeves update) in R programme package "optim". The log likelihood function of the new distribution can be written as follows:

$$\log L(\boldsymbol{\Theta})=n\log(2\varphi\Omega\rho y)-\sum_{i=1}^{n}y_i^2+(1-\rho)\log\sum_{i=1}^{n}(1-e^{-y_i^2})+(\Omega-1)\log\sum_{i=1}^{n}[\Big(1-\left(1-e^{-y_i^2)}\right)^{\varphi}\Big]-2\log\sum_{i=1}^{n}\{1-1-\bar{\rho}\,[\left(1-e^{-y_i^2}\right)^{\varphi}]^{\Omega} \tag{22}$$

$$\frac{\partial}{\partial\varphi}\log L(\Theta)=\frac{n}{\varphi}+\ln(n+\sum_{i=1}^{n}\left(1-e^{-y_i^2}\right)+\frac{(\Omega-1)\sum_{i=1}^{n}\left[-\left(1-e^{-y_i^2}\right)^{\varphi}\ln\left(1-e^{-y_i^2}\right)\right)}{n+\sum_{i=1}^{n}[-\left(1-e^{-y_i^2}\right)^{\varphi}}-\frac{2\sum_{i=1}^{n}\frac{(1-\left(1-e^{-y_i^2}\right)^{\varphi})^{\Omega}\Omega\left(1-e^{-y_i^2}\right)^{\varphi}\ln\left(1-e^{-y_i^2}\right)(\rho-1)}{-1+\left(1-e^{-y_i^2}\right)^{\varphi}}}{n+\sum_{i=1}^{n}(\rho-1)\left(1-\left(1-e^{-y_i^2}\right)^{\varphi}\right)^{\Omega}}=0 \tag{23}$$

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$$\frac{\partial}{\partial \Omega}\log L(\Theta) = \frac{n}{\Omega} + \ln(n + \sum_{i=1}^{n} - \left(1 - e^{-y_i^2)}\right)^{\varphi}) - 2\,\frac{\sum_{i=1}^{n}\left(1-\left(1-e^{-y_i^2}\right)^{\varphi})^{\Omega}\ln\left(1-(1-e^{-y_i^2}\right)^{\varphi}\right)(\rho-1)}{n+\sum_{i=1}^{n}(\rho-1)\left(1-\left(1-e^{-y_i^2}\right)^{\varphi}\right)^{\Omega}} = 0 \quad (24)$$

$$\frac{\partial}{\partial \rho}\log L(\Theta) = \frac{\mathbf{n}}{\boldsymbol{\rho}} - \frac{\sum_{i=1}^{n}(1-\left(1-e^{-y_i^2}\right)^{\varphi})^{\Omega}}{n+\sum_{i=1}^{n}(\rho-1)\left(1-\left(1-e^{-y_i^2}\right)^{\varphi}\right)^{\Omega}\rho-\left(1-\left(1-e^{-y_i^2}\right)^{\varphi}\right)^{\Omega}}=0 \quad (25)$$

The aforementioned equations cannot be solved analytically. The iterative method as the CG method must be used. So, the latter (conjugate gradient method) used the default function of R program, in which called "optim" function with " Fletcher–Reeves update [14].

$$\beta_k^{FR} = \frac{g_{k+1}^{T} g_{k+1}}{g_k^{T} g_k} \quad (26)$$

" to obtain the MLEs of φ, Ω and ρ by means of conjugate gradient method. In addition, the initial value of CG method was suggested as follows:

$$\varphi^{0} = \frac{0.05}{\log(\overline{y}) - mean(\log(y))}\ , \Omega^{(0)} = 0.001 \ \ and\ \rho^{(0)} = 0.001$$

Where φ^{0} is MLE of the Marshall – Olkin Exponentiated Burr type X distribution (MOEBX).

4. Simulation Study:

In this algorithm, the random sample of size (n) from MOEBX by using the quantile function (17). Different sample size= 20, 50, 100, 200, 500 and 1000 are employed to achieve simulation study, and three different sets of the parameters (φ, Ω and ρ) and the values are, Set 1= (4,1,1), Set 2= (1,2,1) and Set 3= (3,5,1). The process is repeated 1000 times. The Average E, bias and MSE are presented in Table 1.

In Table 1, the results of the Average E, bias and MSE values of parameters are presented for five sample sizes. According to the results in Table 1, one can see that when the sample size is increased the Average Es, are close to the real values. Also, the MSEs decrease toward zero as the sample size n increases. Based on the simulation study we can conclude that the maximum likelihood estimators are appropriate for estimating the MOEBX parameters.

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Table 1: Average of MLEs (Average E), Bias and Mean Square Errors (MSE) for different parameter values

Set1	n	φ=4			$\Omega = 1$			$\rho = 1$		
		Average E	**Bias**	**MSE**	**Average E**	**Bias**	**MSE**	**Average E**	**Bias**	**MSE**
	20	4.580	0.580	3.542	1.168	0.168	0.625	1.534	0.534	2.541
	50	4.215	0.215	1.452	1.063	0.063	0.096	1.313	0.313	1.209
	100	4.105	0.105	0.734	1.007	0.007	0.044	1.128	0.128	0.478
	200	4.052	0.052	0.366	1.013	0.013	0.024	1.074	0.074	0.192
	500	4.034	0.034	0.142	1.004	0.004	0.010	1.028	0.028	0.071
	1000	4.008	0.008	0.076	0.998	-0.002	0.005	1.010	0.010	0.033
Set2	**n**	φ=1			$\Omega = 2$			$\rho = 1$		
		Average E	**Bias**	**MSE**	**Average E**	**Bias**	**MSE**	**Avera ge E**	**Bias**	**MSE**
	20	1.171	0.171	0.227	2.268	0.268	0.885	1.283	0.283	1.425
	50	1.082	0.082	0.082	2.097	0.097	0.338	1.178	0.178	0.716
	100	1.029	0.029	0.039	2.054	0.054	0.164	1.141	0.141	0.447
	200	1.016	0.016	0.020	2.030	0.030	0.078	1.076	0.076	0.210
	500	1.003	0.003	0.009	2.007	0.007	0.031	1.043	0.043	0.095
	1000	1.003	0.003	0.005	2.006	0.006	0.018	1.023	0.023	0.046
Set3	**n**	φ=3			$\Omega = 5$			$\rho = 1$		
		Average E	**Bias**	**MSE**	**Average E**	**Bias**	**MSE**	**Avera ge E**	**Bias**	**MSE**
	20	3.354	0.354	1.394	5.891	0.891	9.620	1.582	0.582	3.873
	50	3.173	0.173	0.694	5.365	0.365	2.620	1.351	0.351	1.595
	100	3.052	0.052	0.311	5.113	0.113	0.970	1.226	0.226	0.764
	200	3.013	0.013	0.163	5.051	0.051	0.533	1.140	0.140	0.330
	500	3.000	0.000	0.067	5.029	0.029	0.214	1.063	0.063	0.124
	1000	3.002	0.002	0.036	5.023	0.023	0.099	1.033	0.033	0.062

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5. Applications:

The MOEBX distribution is applied to a real life dataset and its performance compared with other compound distributions like Gamma Burr X, Beta Burr X, Weibull Burr X, Exponentiated Burr X and Burr X distributions. The data set used relates to the strengths of 1.5cm glass fibres obtained from [15, 16,4 and 12]. The R software was used to compute the Negative log-likelihood (NLL) value, Maximum likelihood estimates, by means of conjugate gradient method in unconstrained optimization. Akaike Information Criteria (AIC), CAIC, Bayesian Information Criteria (BIC) and HQIC. The distribution that has the lowest value of these criteria is adjudged the best distribution. The result of the analysis is provided in Table 2.

Table 2: Table of Results.

Distributions	NLL	AIC	CAIC	BIC	HQIC
Gamma Burr X	21.63882	49.27765	49.68443	55.70705	51.80636
Beta Burr X	14.88933	37.77866	38.46831	46.3512	41.15028
Weibull Burr X	15.92427	39.84854	40.5382	48.42108	43.22016
Marshal-Olkin Exponentiated Burr X	12.80504	31.61007	32.01685	38.03948	34.13879
Burr X	23.92875	51.85751	52.05751	56.14378	53.54332
Exponentiated Burr X	23.75491	51.50983	51.70983	55.7961	53.19564

From Table 2, the MOEBX distribution has the lowest value of Negative log-likelihood value, AIC, CAIC, BIC and HQIC, therefore, it is selected as the best out of the distributions considered. The parameter estimates and their associated standard error are provided in Table 3. The histogram of the dataset with all the fitted models is provided in Fig. 3 as a histogram of the fitted models.

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Table 3: Parameter Estimates.

Distributions	Estimates (with Standard Error in Parentheses)
Gamma Burr X	$\alpha = 4.8799605(4.361208)$ $\beta = 0.5606014(0.212984)$ $\theta = 6.5753990(1.689020)$
Beta Burr X	$\alpha = 0.3984152(0.2435249)$ $\beta = 34.6863506(39.6354339)$ $\lambda = 0.5356868(0.1602211)$ $\theta = 7.6109310(5.0002162)$
Weibull Burr X	$\alpha = 105.9769741(180.6594055)$ $\beta = 0.3332802(0.1844864)$ $\lambda = 0.3193555(0.1238970)$ $\theta = 9.6566471(5.3671087)$
Marshal-Olkin Exponentiated Burr X	$\rho = \alpha = 0.02452189(0.02951012)$ $\varphi = \beta = 2.05967645(0.27325360)$ $\Omega = \theta = 1.89290726(1.25566039)$
Burr X	$\lambda = 0.9868927(0.05395035)$ $\theta = 5.4864421(1.18512175)$
Exponentiated Burr X	$\varphi = 6.183078(1.0466122)$ $\Omega = 1.121019(0.1995896)$

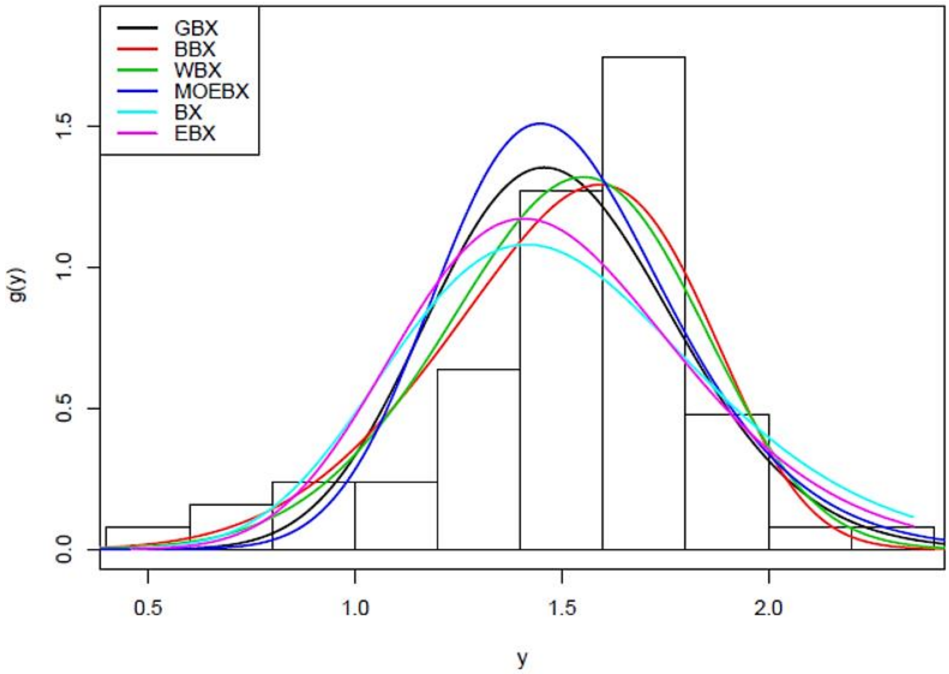

Fig. 3: Histogram of the fitted models.

The plots in Fig. 3 also confirm the results in Table 2.

6. Conclusion:

The numerical experiment confirmed that the CG method was effective in the computational solving of unconstrained optimization problems, in which used to estimate in simulation section. The MOEBX distribution has been successfully studied and its various statistical properties have been established. The distribution is flexible and versatile; it performs better than the Gamma Burr X, Beta Burr X, Weibull Burr X, Exponentiated Burr X and Burr X distributions. The estimates of the parameters are quite stable and close to the true values as we increase the sample size. We hope that this newly introduced distribution would gain wider attention in modelling real life events in engineering, finance, medicine and so on.

References

[1] W. Irving, Burr, "***Cumulative frequency functions***", The Annals of mathematical statistics 13(2), 215 (1942).

[2] Mundher Khaleel, Abdullah, Noor Akma Ibrahim, Mahendran Shitan, and Faton Merovci. "***New extension of Burr type X distribution properties with application***" Journal of King Saud University-Science, 30(4), 450 (2018).

[3] J. G.S., urles and W. J. Padgett. "***Inference for reliability and stress-strength for a scaled Burr type X distribution***", Lifetime Data Analysis, 7(2), 187 (2001).

[4] Faton, Merovci, Mundher Abdullah Khaleel, Noor Akma Ibrahim, and Mahendran Shitan, "***The beta Burr type X distribution properties with application***", SpringerPlus 5(1), 697 (2016).

[5] Mundher Khaleel, Abdullah, Noor Akma Ibrahim, Mahendran Shitan, and Faton Merovci, "***Some properties of Gamma Burr type X distribution with application***", In AIP Conference proceedings, 1739(1), 020087, (2016).

[6] A.O. Adejumo, E. Pelumi Oguntunde, and OA Odetunmibi. "***Some Basic Statistical Properties of the Transmuted Burr X Distribution***", The Pacific Journal of Science and Technology, 18(1),107 (2017).

[7] Albert W. Marshall, and Ingram Olkin., "***A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families***" Biometrika 84(3) ,641 (1997).

[8] Gauss M., Cordeiro, Morad Alizadeh, Abraao DC Nascimento, and Mahdi Rasekhi. "***The Exponentiated Gompertz Generated Family of Distributions: Properties and Applications***", Chilean Journal of Statistics (ChJS), 7(2), (2016).

[9] Adebowale Olusola Adejumo, Oguntunde, Emmanuel Pelumi, and Enahoro Alfred Owoloko., "***Exponential inverse exponential (EIE) distribution with applications to lifetime data***", Asian Journal of Scientific Research, 10, 169 (2017).

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 14, Issue 3, September 2019 , pp. (33-49)
ISSN: 1992-0849 (Print), 2616-6801 (Online)

[10] Morad , Alizadeh , Gauss M. Cordeiro, Luis Gustavo Bastos Pinho, and Indranil Ghosh. "***The Gompertz-G family of distributions***", Journal of Statistical Theory and Practice, 11(1) ,179 (2017).

[11] A. Enahoro, Owoloko., E. Pelumi Oguntunde, and Adebowale O. Adejumo. "***Performance rating of the transmuted exponential distribution: an analytical approach***" SpringerPlus 4(1), 818 (2015).

[12] E. Pelumi, Oguntunde, Mundher Khaleel, A., Mohammed T. Ahmed, Adebowale O. Adejumo, and Oluwole A. Odetunmibi., "***A New Generalization of the Lomax Distribution with Increasing, Decreasing, and Constant Failure Rate***" Modelling and Simulation in Engineering 2017,6 (2017).

[13] Saralees,Nadarajah, and Kotz, Samuel., "***The exponentiated type distributions***", Acta Applicandae Mathematica, 92(2), 97 (2006).

[14] Fletcher, Reeves, and M. Colin Reeves., "***Function minimization by conjugate gradients***", The computer journal, 7(2), 149 (1964).

[15] L. Richard, Smith., and J. C. Naylor. "***A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution.***" Journal of the Royal Statistical Society: Series C (Applied Statistics) 36(3), 358 (1987).

[16] Marcelo, Bourguignon , Rodrigo B. Silva, and Gauss M. Cordeiro. "***The Weibull-G family of probability distributions***", Journal of Data Science, 12(1), 53 (2014).