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Finite Basin's Area Fractal via Complex Newton's Method

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ABSTRACT

In this study, we explain that when we applied Newton’s method on $F(z) = P(z)e^{Q(z)}$, the basins of roots have finite area when $n \geq 3$, where $P(z) = z$ and $Q(z) = z^n$. Using MATLAB we obtained nice fractals in order to prove the finite basins area when $n \geq 3$. The basins for each root in Julia set is finite area of $n \geq 3$. If n is even means the attracting petals of the fixed points zero and infinity are symmetric about x and y axes.

Keywords: Fixed Point, Rational Function, Julia Set, Newton Method, Basins, Exponential Function.

كسيريات منتهية المساحة بواسطة طريقة نيوتن المركبة

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الملخص

في هذه الدراسة اوضحنا عند تطبيق طريق نيوتن المركبة على الدالة المركبة الأسية فان احواض الجذور تكون ذات مساحة منتهية.

الكلمات المفتاحية: النقطة الثابتة , الدالة النسبية ,مجموعة جوليا ,طريقة نيوتن , الأحواض ,الدالة الأسية.

1. Introduction

The study of functions of a complex variable has enormous practical use in applied mathematics as well as in other branches of science and engineering. The impetus to study proper the complex numbers first arose in the 16^{th} century when algebraic solutions for the roots of cubic and quadratic polynomials were discovered by Italian mathematicians. It was soon realized that these formulas, even if one was only interested in real solutions, sometimes required the manipulation of square roots of negative numbers [1, 2]. The first modern study of iteration was due to Ernst Schroder, a Gymnasium teacher in Germany who published two papers in Mathematische Annalen in 1870–71. Although his treatment is not very rigorous, he was the first to suggest the use of conjugation as a means to studying the dynamical behavior of an analytic function near a fixed point [3, 4]. Fractals is a new branch of mathematics and art. Fractal geometry, largely inspired by Benoit Mandelbrot during the sixties and seventies, is one of the great advances in mathematics for two thousand years [5]. Given the rich and diverse power of developments in mathematics and its applications many scientists have found that the fractal geometry is a powerful tool for uncovering secrets from a wide variety of systems and solving important problems in applied science [6, 7]. Also almost all studies of fractals, are coming out from iterations of rational functions in the complex domain [8]. Julia set of a rational function is defined as the set of all repelling periodic points and Fatou set is the opposite of Julia set. So each repelling points belonging to Julia and all attracting fixed point of the rational function belonging to Fatou set [9]. The Fatou Flower theorem provides an analytic description of the dynamics around a rationally indifferent fixed point. Therefore, the degree of the exponent polynomial completely determines the number of petals at infinity. Newton's method is known and introduced in calculation for finding the roots of functions when analytical methods has fail. This method is better, if the initial supposition is close to the real root, iterations will converge very fast to the root. The dynamics of Newton's method in the complex plane, provides exciting of fractals which depend on what kind of functions we used. However, Newton's method for finding solutions to equations leads to some fantastic images when it is applied to complex functions, that which called a basin of attraction is defined to be the set of all points that converge to the same root. And the connected component of attraction basins which containing the root of the basin is called the immediate basin of attraction. The main results of this paper focus on the areas of the attraction basins [10-15].

2. Rational Functions Iterations

Let T and S are complex polynomials, then the rational map is knew by,

$$R(z) = \frac{T(z)}{S(z)}$$

$R(z)$ is a rational map of a degree d bigger than or equal 2 on the Riemann sphere, the n^{th} iterates of R are defined by R^n where is $R^n = R o R o R....o R$. The important problem of $R(z)$ dynamics is to know the behavior of high iterates $R^n(z) = R o R^{n-1}(z)$ [1]. z is a fixed point of $R(z)$ if $R(z) = z$ [1]. The derivative $R'(z)$ is defined by the number $\lambda = R'(z)$ and λ is called the multiplier of R at z. We able to classify the fixed point of the rational map according to λ, as follows, z is attracting, if $|\lambda| < 1$, z is repelling, if $|\lambda| > 1$, z is neutral, if $|\lambda| = 1$, z is a

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super attracting fixed point, if $|\lambda| = 0$. The orbit $\{z_0, z_1, ..., z_n = z_0\}$ is called a cycle [1]. We classify the cycle as (super) attracting, repelling and rationally indifferent or irrationally indifferent according to the type of the fixed point of R^n. For example the cycle is attracting if and only if $|R'^n(z)| < 1$ [2]. For each rational function $R(z)$ it able to conjugate $R(z)$ with the conversion $z \to 1/z$. Therefore, when $z = \infty$ the behavior of $R(z)$ is the same behavior of $1/S(1/z)$ at 0 [3]. The two polynomials $T(z)$ and $S(z)$ allow us to find the poles and zeros of the rational function, zeroes is the values for z, where $T(z) = 0$ and the pole is the value of z, where $S(z) = 0$. The critical points of a rational function $R(z)$ are those where $R'(z)$ vanishes [4].

3. An Attracting Petals and Repelling Petals

"Suppose M is a neighborhood U of the origin and [p] is an attracting petal for M at fixed point if $M(\overline{p}) \subset [\overline{p}] \cup \{0\}$ and $\bigcap_{n \geq 0} M^n(\overline{p}) = \{0\}$. A repelling petal [p] is an attracting petal for M^{-1} which exists locally since $M^{-1} = 1$. M^{-1} denotes the branch of the inverse of M fixing the origin" [5,6].

4. Leu-Fatuo Flower Theorem

"If M is a holomorphic map of the form $M(z) = z + cz^{n+1} + O(z^{n+2}), c \neq 0, n \geq 1$. Defined in some neighborhood of the origin with $\lambda = 1$. Then (0) is a parabolic fixed point and there are n attracting petals and n repelling petals for M at zero. Moreover, these petals alternate with one another" [6]. For simplicity of Leu-Fatuo Flowers theorem [6]. It will assume that (c =1) when λ=1, then there exist respectively attracting and repelling direction rays along which orbits tend to zero and ∞. Leau–Fatou Flower application gives the information of local dynamics for $N(z)$ near infinity through the study of M near the origin. Therefore, $M(v) = v + cv^{n+1} + Ov^{n+2}$.

Where $c = \dfrac{1}{n}$.

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5. Newton's Method Dynamics on $F = Pe^{Q}$

Let P, Q are complex polynomial, such that $P:\mathbb{C}\to\mathbb{C}$ of degree m, $m \ge 0$, and $Q:\mathbb{C}\to\mathbb{C}$ of degree n, $(n \ge 1)$.When we applied the complex Newton's method on the exponential function $F(z) = P(z)e^{Q(z)}$ we obtain**,**

$$N(z) = z - \frac{F(z)}{F'(z)} = z - \frac{P(z)e^{Q(z)}}{P'(z)e^{Q(z)} + Q'(z)e^{Q(z)}P(z)}$$

$$= z - \frac{P(z)}{P'(z) + Q'(z)P(z)}$$

Will get rational map $N(z)$, as $N(z):\mathbb{C}_\infty \to \mathbb{C}_\infty$. By rational map $N(z)$ we can examine the resulting dynamics of exponential function $F(z)$ Recall that, the fixed point of $N(z)$ coincide with roots of F [6]. We can determine the nature of the fix points by $N'(z)$ as follows, If the fix point, $z \ne a$, as a multiple root, then

$$N'(z) = \frac{F(z)F''(z)}{(F'(z))^2}$$

Then, the simple root for F is super attracting fixed point for $N(z)$.

If $z = a$ is afixed point then a is a multiple root of $F(z)$ with multiplicity. The derivative of $N(z)$ is given by $N'(z) = \frac{m-1}{m}$, m is multiplicity of a. If $z = a$, is a critical point of F, then $N(z)$ has a pole at a if and only if a is not a root of F. If a is a critical point and not a root of F, then a will be send to infinity under single iteration of $N(z)$ [6].

The speed of the convergence to a root depends on the multiplicity, therefore, there is an inverse relationship between them, and the higher multiplicity gives slower converge. There is an inverse relationship between them, and the higher multiplicity gives slower converge [6].

Proposition 1: "Infinity is a parabolic fixed point, with multiplier equal to 1, for Newton's method applied to $F(z) = P(z)e^{Q(z)}$, where P and Q are complex polynomials, P is not identically zero and Q is not constant"[6].

Proposition 2: There is just n repelling petals, n attracting petals for the neutral fixed point infinity if Q has degree of n "[6].

6. Basin's

Newton's method for solving an equations leads to some fantastic images when it's applied to functions in complex plane. That which called a basin of attraction is defined to be the set of all points that converge to the same root (ξ^*) when $(n \to \infty)$ and we denoted it by $A(\xi^*)$, here (ξ^*) is roots of $F(z) = 0$. Since (ξ^*) is the super- attracting fixed point of the Newton's method then $A(\xi^*)$ is an open region including (ξ^*)[7]. So coloring each basin of attraction by different color, the boundaries between the basins are defined Julia set, in other words we can define Julia set by $J_F = \partial A(\xi_1^*) = ... = \partial A(\xi_n^*)$. These boundary points are points that do not converge in any of Newton's method and form fractal image. The connected component of attraction basins $A(\xi^*)$ which containing (ξ^*) of the basin is called the immediate basin of (ξ^*) [8, 9].

7. Basin's Area Theorem

"If Newton's method is applied to $F = Pe^Q$, where P and Q are are polynomials in the □ plane, P is not identically zero and Q is non-constant with degree of $Q \geq 3$, then the area of the attractive basin of a root of F has finite area" [6].

8. Basin's Fractal

In this work, we will use the functions that shape $F(z) = P(z)e^{Q(z)}$ (which Haruta [6] used it in his studies) to generate fractal on z-plane with different value of (n = 3, 4, 6- 12) in some examples to check up the basins area and showing of our results. We simulate our methods with MATLAB. So, we used $| N^k(z) | < 0.001$ and $|k| \leq 50$, which k is the iteration number of Newton's method [10-13].

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9. Experiments and Results

Example (1): When $n=3$, $F(z)=ze^{z^3}$. Therefore, $N(z)$ of $F(z)=ze^{z^3}$ gives,

$$N(z)=z-\frac{ze^{z^3}}{(ze^{z^3})'}=z-\frac{z}{1+3z^3}.$$

When $z=0$, we have $N(0)=0$. Since, $N(0)=0$, then zero is a simple root of F. To determine the nature of zero by the derivative of $N(z)$ as follows, $N'(z)=1-\frac{1-6z^3}{(1+3z^3)^2}$. When $z=0$, we have $N'(0)=0$. Since, $N'(0)=0$, then, zero is super attracting fixed point of $N(z)$. Since, $\lim_{z\to\infty} N(z)=\infty$, then, ∞ is a fixed point of $N(z)$. Conjugate $N(z)$, by $g(z)=\frac{1}{z}$ to M from infinity *to* zero as, $M(v)=v+\frac{1}{3}v^4$. When v = 0, $M'(v)=1$, since, $M'(v)=1$, then, ∞ is a parabolic fixed point of $N(z)$.

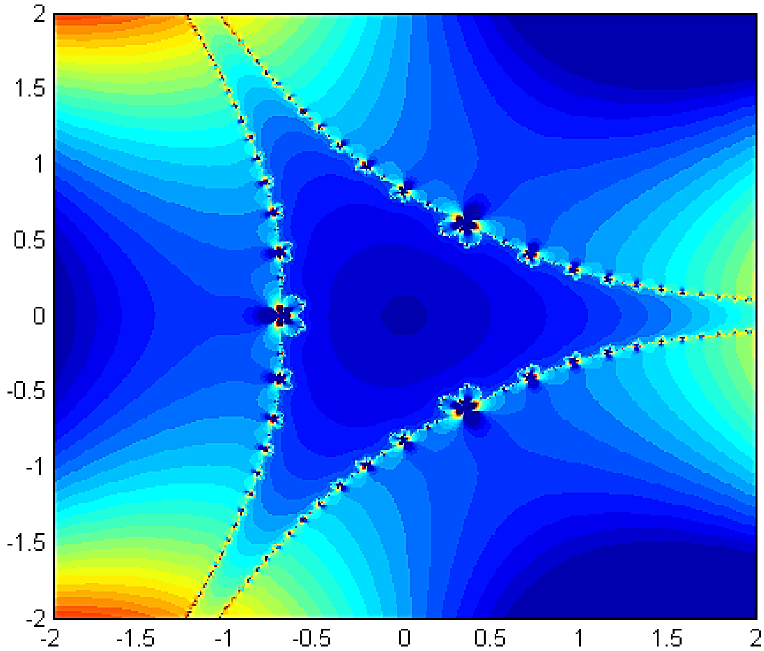

Fig. (1): Basins Fractal of $F(z)=ze^{z^3}$ for $n=3$

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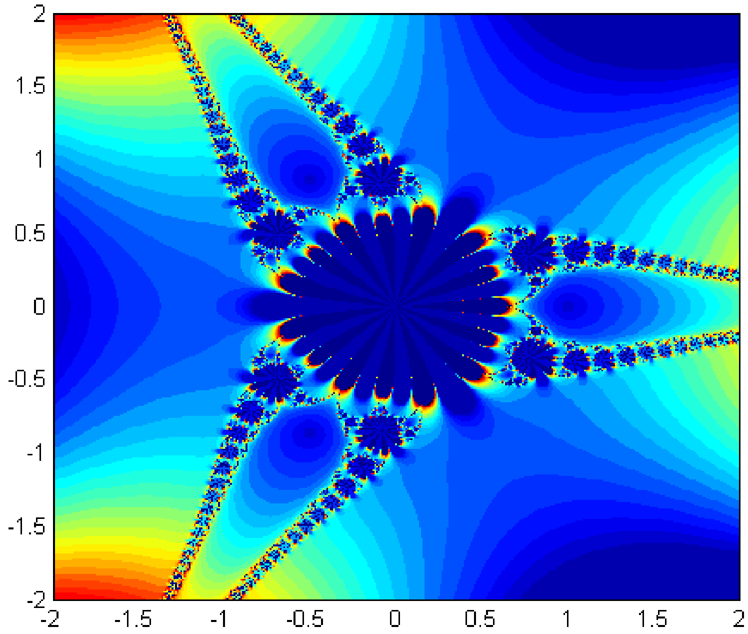

Fig. (2): Basins Fractal of $F(z) = (z^3 - 1)e^{z^3}$ for $n = 3$

Example (2): When $n = 6$, $F(z) = ze^{z^6}$. Therefore, $N(z)$ of $F(z) = ze^{z^6}$ is,

$$N(z) = z - \frac{ze^{z^6}}{(ze^{z^6})'} = z - \frac{z}{1+6z^6}.$$

Zero is a simple root of F. So, the derivative of $N_F(z)$ is $N'(z) = 1 - \frac{1-30z^6}{(1+6z^6)^2}$. Then, zero is super attracting fixed point of $N(z)$. Since $\lim_{z\to\infty} N(z) = \infty$, then ∞ is a parabolic fixed point of $N(z)$. The series expansion of M is, $M(v) = v + \frac{1}{6}v^7$.

So, $M'(v)$ is $M'(v) = 1 + \frac{7}{6}v^6$. When $v = 0, M'(0) = 1$, then, ∞ is a parabolic fixed point of $N(z)$. Since deg(Q)= 6, then there are six attracting and six repelling petals for $N(z)$ at ∞. Also the basin is lied in Fatuo set. The parabolic point lies on the boundary of the basin and in the Julia set. The basin has finite area. Since, n is even the attracting petals are symmetric about x and y axes.

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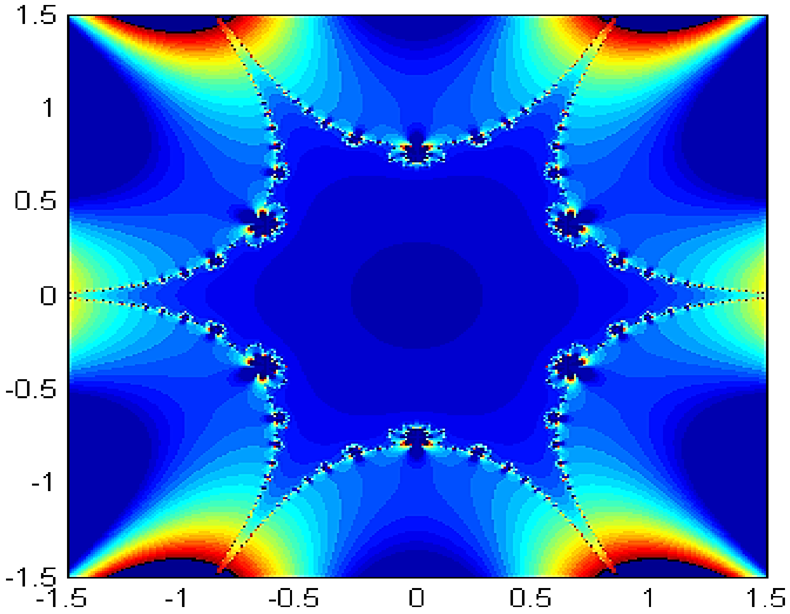

Fig. (3): Basins Fractal of $F(z) = ze^{z^6}$ for $n = 6$

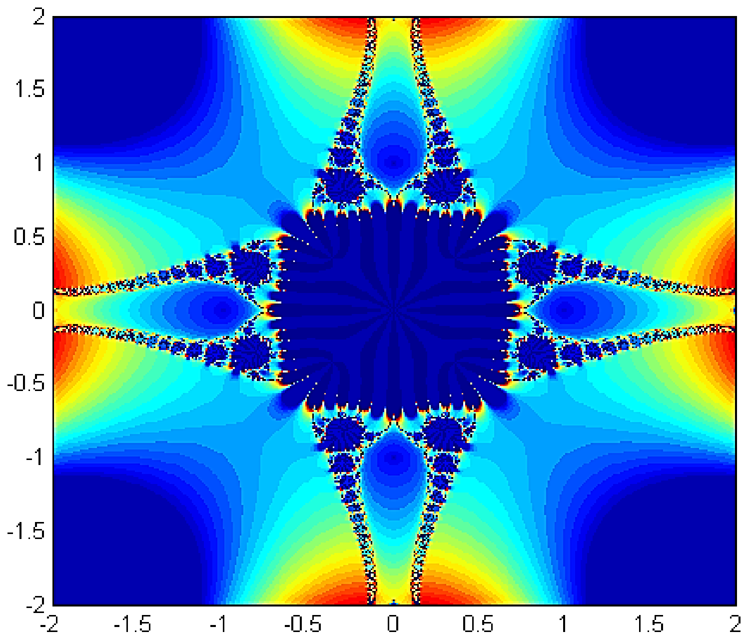

Fig. (4): Basins Fractal of $F(z) = (z^4 - 1)(z^4 - 1)e^{z^4}$

Example (3): Some finite basins area fractals of $F(z) = ze^{z^n}$ for different n.

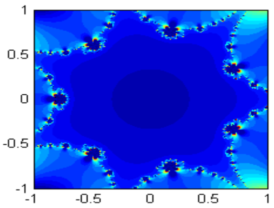

Fig. (5) $n = 7$

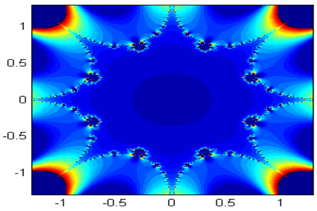

Fig. (6) $n = 8$

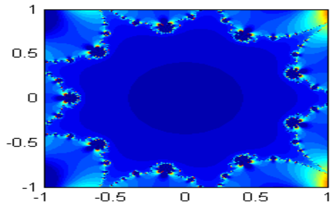

Fig. (7) $n = 9$

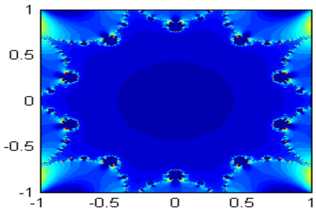

Fig. (8) $n = 10$

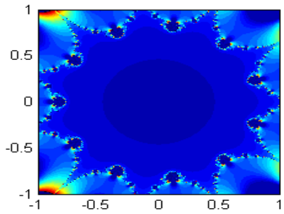

Fig. (9) $n = 11$

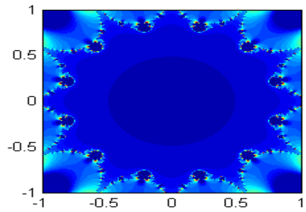

Fig. (10) $n = 11$

10. Conclusion

In this paper, we gave examples fractal to determine the type of basins area when $n \geq 3$ by using complex Newton method on exponential functions " $F(z) = P(z)e^{Q(z)}$, where $P(z) = z$ and $Q(z) = z^n$. We simulate our method with MATLAB. Therefore, when $n \geq 3$ we obtain finite area of complex Newton's basin. Also, when we change $P(z) = z$ in the complex exponential function to another polynomial we get also the same results when $n \geq 3$, that means the area of the basin depend on the deg(Q) not on the $P(z)$. The basin lie in Julia set for all n. The basins for each root in Julia set is finite area of $n \geq 3$. If n is even means the attracting petals of the fixed points zero and infinity are symmetric about x and y axes. We also added Color Palette's Descriptions table of the finite area.

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